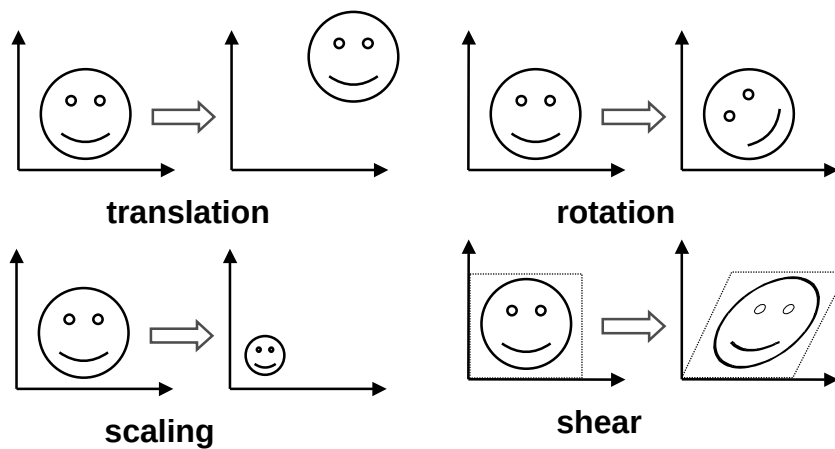

Geometry Review II

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Transformations

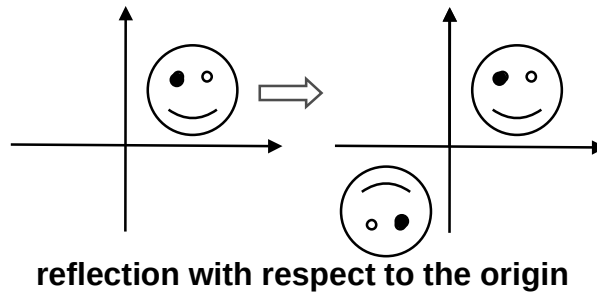
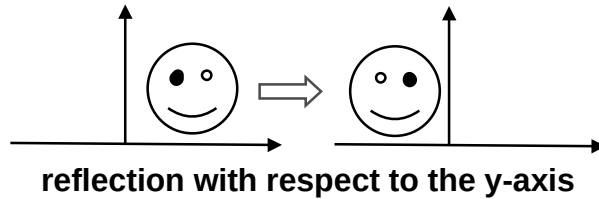
Examples of transformations:



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Transformations

More examples:



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Transformations

Linear transformations: take straight lines to straight lines.

All of the examples are linear.

Affine transformations: take parallel lines to parallel lines.

All of the examples are affine,

an example of linear non-affine is perspective projection.

Orthogonal transformations: preserve distances, move all objects as rigid bodies.

rotation, translation and reflections are affine.

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Transformations and matrices

Any affine transformation can be written as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad p' = Ap$$

Images of basis vectors under affine transformations:

$$\begin{aligned} e_x &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & (\text{column form of writing vectors}) \\ e_y &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad \begin{aligned} Ae_x &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} & Ae_y = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \end{aligned}$$

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Transformations and matrices

Matrices of some transformations:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ shear} \quad \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \text{ scale by factor } s$$

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \text{ rotation}$$

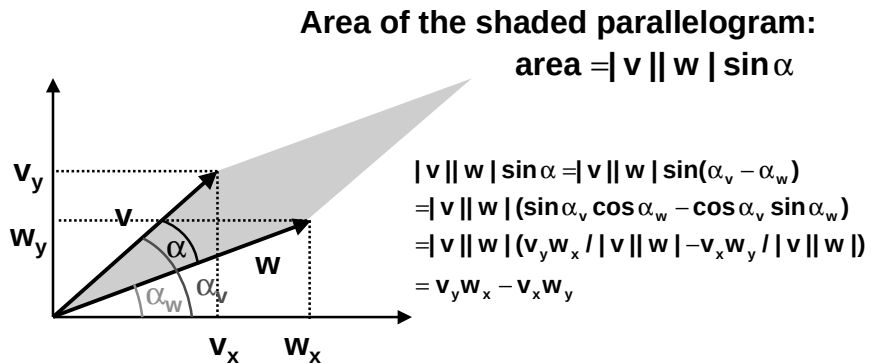
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ reflection with respect to the origin}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ reflection with respect to } y\text{-axis}$$

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Determinant

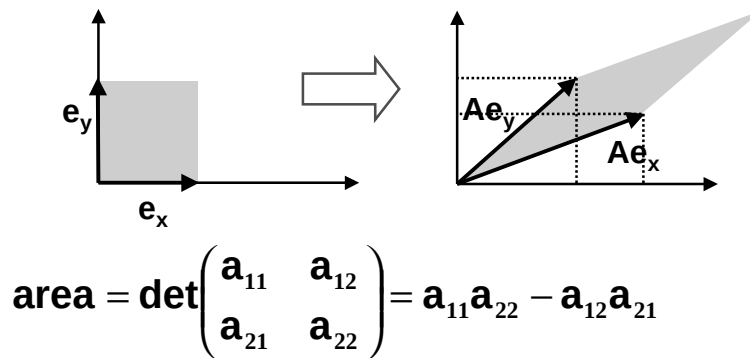
Area of a parallelogram



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Determinant

Determinant of a transformation matrix = area of the image of a unit square.



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3D vectors

Same as 2D (directed line segments with position ignored), but we have different properties.

In 2D, the vector perpendicular to a given vector is unique (up to a scale).

In 3D, it is not.

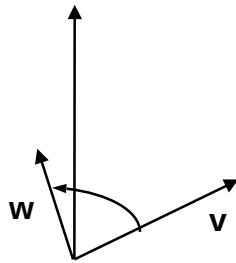
Two 3D vectors in 3D can be multiplied to get a vector (vector or cross product).

Dot product works the same way, but the coordinate expression is

$$(v, w) = v_x w_x + v_y w_y + v_z w_z$$

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Vector (cross) product



$v \times w$ has length $|v||w| \sin \alpha$

= area of the parallelogram with two sides given by v and w , and is perpendicular to the plane of v and w .

$(v + w) \times u = v \times u + w \times u$ Direction (up or down) is determined by the right-hand rule.

$(cv) \times w = c(v \times w)$

$$\boxed{v \times w = -w \times v}$$

unlike a product of numbers or dot product, vector product is not commutative!

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Vector product

Coordinate expressions

$\mathbf{v} \times \mathbf{w}$ is perpendicular to $\mathbf{v}$, and $\mathbf{w}$:

$$(\mathbf{u}, \mathbf{v}) = 0 \quad (\mathbf{u}, \mathbf{w}) = 0$$

the length of $\mathbf{u}$ is $|\mathbf{v}||\mathbf{w}|\sin\alpha$:

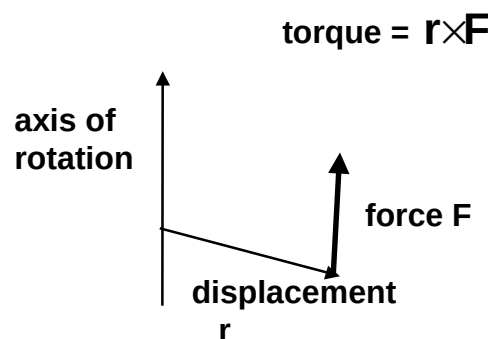
$$\begin{aligned} (\mathbf{u}, \mathbf{u}) &= |\mathbf{v}|^2 |\mathbf{w}|^2 \sin^2 \alpha = |\mathbf{v}|^2 |\mathbf{w}|^2 (1 - \cos^2 \alpha) \\ &= |\mathbf{v}| |\mathbf{w}| (|\mathbf{v}||\mathbf{w}| - (\mathbf{v}, \mathbf{w})) \end{aligned}$$

Solve three equations for u_x, u_y, u_z

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Vector product

Physical interpretation: torque



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Vector product

Coordinate expression:

$$\det \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} = \left[\det \begin{bmatrix} v_y & v_z \\ w_y & w_z \end{bmatrix}, -\det \begin{bmatrix} v_x & v_z \\ w_x & w_z \end{bmatrix}, \det \begin{bmatrix} v_x & v_y \\ w_x & w_y \end{bmatrix} \right]$$

Notice that if $v_z=w_z=0$, that is, vectors are 2D, the cross product has only one nonzero component (z) and its length is the determinant

$$\det \begin{bmatrix} v_x & v_y \\ w_x & w_y \end{bmatrix}$$

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Vector product

More properties

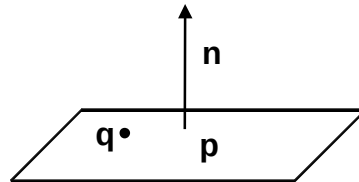
$$(\mathbf{a}, \mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}\mathbf{c}) - \mathbf{c}(\mathbf{a}\mathbf{b})$$

$$(\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}) = (\mathbf{a}, \mathbf{c})(\mathbf{b}, \mathbf{d}) - (\mathbf{b}, \mathbf{c})(\mathbf{a}, \mathbf{d})$$

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Plane equations

implicit equation: $(q-p,n)=0$, exactly like line in 2D!



parametric equation: 2 parameters t_1, t_2

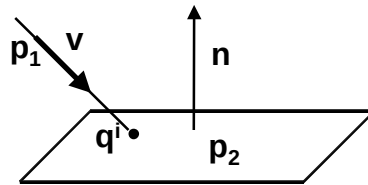
$q(t_1, t_2) = v_1 t_1 + v_2 t_2$, where v_1 and v_2 are two vectors in the plane.

$$v_1 \times v_2 = n$$

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Intersecting a line and a plane

Same old trick: use the parametric equation for the line, implicit for the plane.



$$(p_1 + vt^i - p_2, n) = 0$$

$$t^i = -\frac{(p_1 - p_2, n)}{(v, n)} \quad \text{Do not forget to check for zero in the denominator!}$$

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