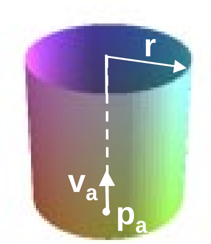
Ray tracing II

Infinite cylinder-ray intersections



Infinite cylinder along y of radius r axis has equation $x^2 + z^2 - r^2 = 0$.
The equation for a more general cylinder of radius r oriented along a line $p_a + v_a t$:
$(q - p_a - (v_a, q - p_a) v_a)^2 - r^2 = 0$
where $q = (x,y,z)$ is a point on the cylinder.

Infinite cylinder-ray intersections

To find intersection points with a ray p + vt, substitute q = p + vt and solve:

$(\mathbf{p} - \mathbf{p_a} + \mathbf{vt} - (\mathbf{v_a},\mathbf{p} - \mathbf{p_a} + \mathbf{vt})\mathbf{v_a})^2 - \mathbf{r}^2 = 0$

reduces to $\mathbf{A}\mathbf{t}^2 + \mathbf{B}\mathbf{t} + \mathbf{C} = 0$

with

$$\mathbf{A} = (\mathbf{v} - (\mathbf{v},\mathbf{v_a})\mathbf{v_a})^2$$
$$\mathbf{B} = 2(\mathbf{v} - (\mathbf{v},\mathbf{v_a})\mathbf{v_a}, \Delta\mathbf{p} - (\Delta\mathbf{p},\mathbf{v_a})\mathbf{v_a})$$
$$\mathbf{C} = (\Delta\mathbf{p} - (\Delta\mathbf{p},\mathbf{v_a})\mathbf{v_a})^2 - \mathbf{r}^2$$

where $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p_a}$

Cylinder caps

A finite cylinder with caps can be constructed as the intersection of an infinite cylinder with a slab between two parallel planes, which are perpendicular to the axis.

To intersect a ray with a cylinder with caps:

- intersect with the infinite cylidner;
- check if the intersection is between the planes;
- intersect with each plane;
- determine if the intersections are inside caps;
- out of all intersections choose the on with minimal t

Cylinder-ray intersections

POV -ray like cylinder with caps : cap centers at p_1 and p_2, radius r.

Infinite cylinder equation: $p_a = p_1$, $v_a = (p_2 - p_1)/| p_2 - p_1|$

The finite cylinder (without caps) is described by equations:

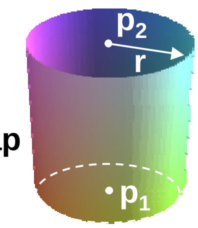
$(q - p_a - (v_a, q - p_a)v_a)^2 - r^2 = 0$ and $(v_a, q - p_1) > 0$ and

$(v_a, q - p_2) < 0$

The equations for caps are:

$(v_a, q - p_1) = 0$, $(q - p_1)^2 < r^2$ bottom cap

$(v_a, q - p_2) = 0$, $(q - p_2)^2 < r^2$ top cap



Cylinder-ray intersections

Algorithm with equations:

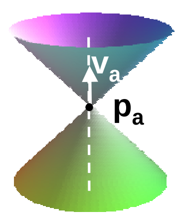
Step 1: Find solutions t_1 and t_2 of $At^2 + Bt + C = 0$

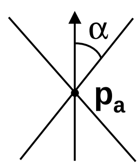
if they exist. Mark as intersection candidates the one(s) that are nonnegative and for which $(v_a, q_i - p_1) > 0$ and $(v_a, q_i - p_2) < 0$, where $q_i = p + v\, t_i$

Step 2: Compute t_3 and t_4, the parameter values for which the ray intersects the upper and lower planes of the caps.
If these intersections exists, mark as intersection candidates those that are nonegative and $(q_3 - p_1)^2 < r^2$ (respectively $(q_4 - p_2)^2 < r^2$).

In the set of candidates, pick the one with min. t.

Infinite cone-ray intersections





Infinite cone along y with apex half-angle α has equation $x^2 + z^2 - y^2 = 0$.
The equation for a more general cone oriented along a line $p_a + v_a t$, with apex at p_a:
$\cos^2\alpha\,(q - p_a - (v_a, q - p_a)v_a)^2 - \sin^2\alpha\,(v_a, q - p_a)^2 = 0$
where $q = (x,y,z)$ is a point on the cone, and v_a is assumed to be of unit length.

Infinite cone-ray intersections

Similar to the case of the cylinder: substitute $q = p+vt$ into the equation, find the coefficients A, B, C of the quadratic equation, solve for t. Denote $\Delta p = p - p_a$.

$$\cos^2\alpha\,(vt + \Delta p - (v_a, vt + \Delta p)v_a)^2 - \sin^2\alpha\,(v_a, vt + \Delta p)^2 = 0$$

$$A = \cos^2\alpha\,(v - (v, v_a)v_a)^2 - \sin^2\alpha\,(v, v_a)^2$$

$$B = 2\cos^2\alpha\,(v - (v, v_a)v_a, \Delta p - (\Delta p, v_a)v_a) - 2\sin^2\alpha\,(v, v_a)(\Delta p, v_a)$$

$$C = \cos^2\alpha\,(\Delta p - (\Delta p, v_a)v_a)^2 - \sin^2\alpha\,(\Delta p, v_a)^2$$

Cone-ray intersections

A finite cone with caps can also be constructed as intersection of an infinite cone with a slab.

Intersections are computed exactly in the same way as for the cylinder, but instead of the quadratic equation for the infinite cylinder the equation for the infinite cone is used, and the caps may have different radii.

Both for cones and cylinders intersections can be computed somewhat more efficiently if we first transform the ray to a coordinate system aligned with the cone (cylinder). This requires extra programming to find such transformation.

Cone-ray intersections

POV-ray cone: cap centers (base point and cap point) at p_1 and p_2, cap radii r_1 and r_2.

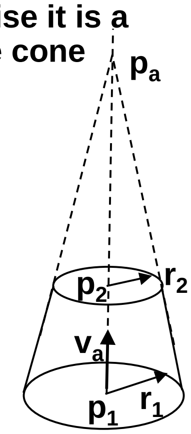
Then, assuming r_1 not equal to r_2 (otherwise it is a cylinder) in the equation of the infinite cone

apex: $p_a = p_1 + r_1(p_2 - p_1)/(r_1 - r_2)$;

axis direction: $v_a = (p_2 - p_1)/|p_2 - p_1|$;

apex half-angle:

$\mathrm{tg}\,\alpha = (r_1 - r_2)/|p_2 - p_1|$



General quadrics

A general quadric has equation

$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

Intersections with general quadrics are computed in a way similar to cones and cylinders: for a ray p+ v t, take $x = p^x + v^x t$, $y = p^y + v^y t$, $y = p^z + v^z t$,

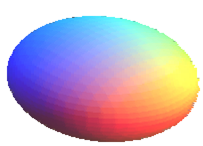
and solve the equation for t; if there are solutions,

take the smaller nonnegative one.

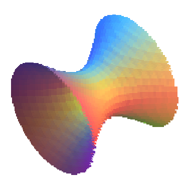
Infinite cones and cylinders are special cases of general quadrics.

General quadrics

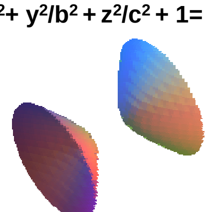
Nondegenerate quadrics



Ellipsoid
$x^2/a^2 + y^2/b^2 + z^2/c^2 + 1 = 0$



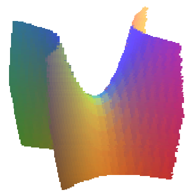
One-sheet hyperboloid
$x^2/a^2 - y^2/b^2 + z^2/c^2 + 1 = 0$



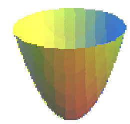
Two-sheet hyperboloid
$x^2/a^2 - y^2/b^2 + z^2/c^2 - 1 = 0$

General quadrics

Nondegenerate quadrics



Hyperbolic parabolid
$x^2/a^2 - z^2/c^2 - 2y = 0$



Elliptic parabolid
$x^2/a^2 + z^2/c^2 - 2y = 0$

cone
$x^2/a^2 - y^2/b^2 + z^2/c^2 = 0$

General quadrics

Degenerate quadrics

- **planes (no quadratic terms),**
- **pairs of parallel planes (e.g. $x^2 - 1 = 0$)**
- **pairs of intersecting planes (e.g. $x^2 - 1 = 0$)**
- **elliptic cylinders (e.g. $x^2+z^2-1=0$)**
- **hyperbolic cylinders (e.g. $x^2-z^2-1=0$)**
- **parabolic cylinders (e.g. $x^2 - z = 0$)**

Possible to get "imaginary" surfaces (that is, with no points)! Example: $x^2 + 1 = 0$

3D transformations

Homogeneous coordinates

regular 3D point to homogeneous:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \longrightarrow \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix}$$

homogeneous point to regular 3D:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ p_w \end{pmatrix} \longrightarrow \begin{pmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{pmatrix}$$

Points vs. vectors

- a translated point is a different point
- a translated vector is the same vector
- in homogeneous coordinates:
  - points have last component nonzero
  - vectors have last component zero

Translation and scaling

Similar to 2D; translation by a vector

$$t = [t_x, t_y, t_z]$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Nonuniform scaling in three directions

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations around coord axes

angle θ, around X axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

around Y axis:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

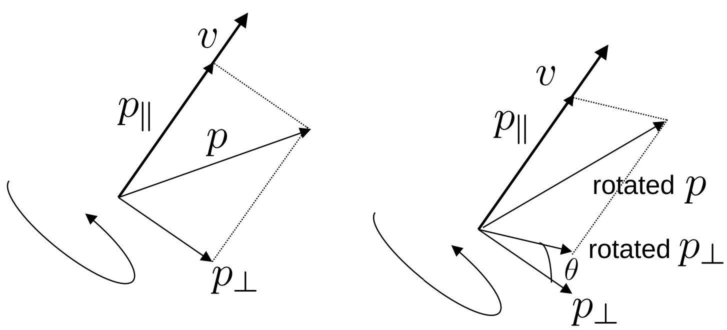
note where the minus is!

around Z axis:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General rotations

Given an axis (a unit vector) and an angle, find the matrix



Only the component perpendicular to axis changes

General rotations

(rotated vectors are denoted with ′)

project p on v: $$p_{||} = (p,v)v$$

the rest of p is the other component: $$p_{\perp} = p - (p,v)v$$

rotate perp. component: $$p'_{\perp} = p_{\perp}\cos\theta + (v \times p_{\perp})\sin\theta$$

add back two components: $$p' = p'_{\perp} + p_{||}$$

Combine everything, using $v \times p_{\perp} = v \times p$ to simplify:

$$\boxed{p' = \cos\theta\ p + (1-\cos\theta)(p,v)v + \sin\theta(v \times p)}$$

General rotations

How do we write all this using matrices?

$$p' = \cos\theta\ p + (1-\cos\theta)(p,v)v + \sin\theta(v \times p)$$

$$(p,v)v = \begin{bmatrix} v_x v_x p_x + v_x v_y p_y + v_x v_z p_z \\ v_y v_x p_x + v_y v_y p_y + v_y v_z p_z \\ v_z v_x p_x + v_z v_y p_y + v_z v_z p_z \end{bmatrix} = \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$(v \times p) = \begin{bmatrix} -v_z p_y + v_y p_z \\ v_z p_x - v_x p_z \\ -v_y p_x + v_x p_y \end{bmatrix} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Final result, the matrix for a general rotation around a by angle θ:

$$\cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

Composition of transformations

- Order matters! (rotation * translation $\neq$ translation * rotation)
- Composition of transformations = matrix multiplication:
  if T is a rotation and S is a scaling, then applying scaling first and rotation second is the same as applying transformation given by the matrix TS (note the order).
- Reversing the order does not work in most cases

Transformation order

- When we write transformations using standard math notation, the closest transformation to the point is applied first:

$$T\ R\ S\ p = T(R(Sp))$$

- first, the object is scaled,
  then rotated,
  then translated
- This is the most common transformation order for an object (scale-rotate-translate)