Homework #2: Subdivision
Due Date: November 11

Problems marked with * are optional problems. Extra credit will be given for these problems.

1. **Polynomial reproduction.** Recall that the four-point interpolating subdivision scheme for curves has coefficients $-1/16$, $9/16$, $9/16$, $-1/16$ for odd points. The limit curves produced by this scheme need not be piecewise polynomials. However, this scheme has a remarkable property relating it to polynomials.

   Prove that if all control points c_i (i is an integer), have coordinates $c_i^x = p^x(i)$, $c_i^y = p^y(i)$, where p^x and p^y are polynomials of degree no greater than 3, then the limit curves generated by the four-point scheme are polynomial curves $(p^x(t), p^y(t))$.

   Note: This is a specific example of a more general property of subdivision: if the curves or surfaces generated by subdivision are C^k-continuous, the scheme should reproduce polynomials of degree at least up to $k+1$.

2. **Convergence of subdivision.** In the class we have seen how to prove that the four-point subdivision scheme converges. In this problem we use the same method to determine when a generalization of the subdivision scheme for quadratic splines. The generalization is defined by the rules

   $$\begin{aligned} x_{2i}^{j+1} &= (1-a)x_i^j + a x_{i+1}^j \\ x_{2i+1}^{j+1} &= a x_i^j + (1-a)x_{i+1}^j \end{aligned}$$

   The subdivision scheme for the quadratic splines corresponds to the value of $a = 1/4$. Assume that $0 < a < 1/2$.

   a. Recall that coordinates of control points on levels j and $j+1$ are related by

   $$x_i^{j+1} = \sum_k s_{2k-i} x_k^j$$

   (x_i^j is one of the x, y coordinates of a control point, a single number, not a 2D point.) Find a formula of the same form relating the *differences* $x_{i+1}^{j+1} - x_i^{j+1}$ to $x_{i+1}^j - x_i^j$ for the scheme defined above.

   b. Deduce from the recursive formula for differences that the differences converge to zero as $j \to \infty$. This is the crucial element of the convergence proof.

   c. Assuming that $\max_i |x_{i+1}^0 - x_i^0|$ is bounded by a constant C, show that the sequence of piecewise-linear approximations generated by subdivision converges pointwise, that is, for any fixed t the linear approximation $|L^j(t) - f(t)|$ converges to zero. $L^j(t)$ is the piecewise-linear approximation to the limit subdivision curve $f(t)$. *Hint.* Estimate the difference $|L^{j+1}(t) - L^j(t)|$ and observe that $|L^j(t) - f(t)| = \sum_{i=j}^{\infty}(L^i(t) - L^{i+1}(t))$. *Note.* The estimates derived in the last part of the proof can be used to estimate the precision of linear approximation $L^j(t)$.

3. **Piecewise-smooth Catmull-Clark.** In the class we have discussed how to modify the subdivision rules for Loop's scheme to add sharp creases to a surface. Describe a similar modification of rules for the Catmull-Clark scheme.

4. **Adaptive subdivision.** One of the attractive features of subdivision is that it is possible to refine curves or surfaces adaptively. For example, we can subdivide more in places where the curve varies a lot, and subdivide less in places where it is almost flat. However, we might need to compute some extra control points which are not necessary to approximate the curve well according to our error criterion, but are required to compute points on finer levels. In this problem we find in a specific case how many extra control points have to be computed.

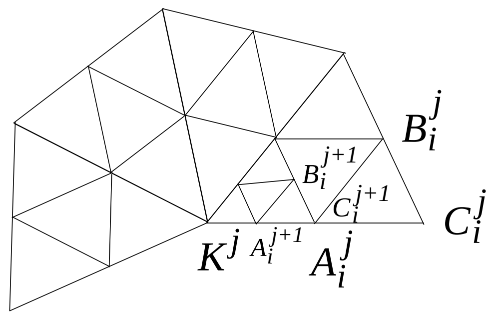
   Suppose we are approximating a degree 5 B-spline curve with 8 control points c_i $i = 0 \ldots 7$. Suppose that we want to subdivide the interval between control points c_3 and c_4 four times, and do not want to subdivide other intervals at all, unless it is required to subdivide the interval of interest. What control points outside the interval between c_3 and c_4 we have to compute? How many additional points have to be computed?

5. **Geometric smoothness and tangent planes.** Show that a surface defined by $x(u, v) = u$, $y(u, v) = v$ $z(u, v) = u^2 \sin 1/u$ for $u \neq 0$, $z(0, v) = 0$, has a tangent plane at all points $u = 0$, but the limit of normals does not exist. This example shows that geometric smoothness is a stronger requirement then simply existence of a tangent plane: in addition, the tangent plane should change continuously.

6. **Subdivision matrix for Loop's scheme.** The subdivision matrix plays an important role in analysis of subdivision. The subdivision matrix can be used to compute tangent vectors of subdivision surfaces, which allow us to compute the exact normal to the surface. Recall that the tangent vectors are given by the coefficients a_1 and a_2 in the decomposition of the vector of control vertices with respect to the eigenbasis of the subdivision matrix.

   It turns out that to compute $\mathbf{a}_1$ and $\mathbf{a}_2$ we do not need the eigenvectors of the whole subdivision matrix, it is sufficient to consider a smaller submatrix, acting on the subset of the control vertices that were used to define subdivision matrix.

   a. The subdivision matrix for Loop's scheme acts on vectors of control vertices in the 2-ring surrounding the vertex that we are interested in. There are 4 types of vertices: K, A, B, C, as shown in the picture below. We are interested only in the submatrix of the subdivision matrix acting on the vector c of control vertices consisting of $K, A_0, A_1, \ldots A_{n-1}$, where n is the degree of the vertex in the center. Write the equations for computing A_i^{j+1} and K^{j+1} using A_i^j, K^j. Assume that $\alpha = 3/8n$, where n is the valence of the central vertex, and $n > 3$. In your equations, use indices modulo n, i.e. assume that $A_{2nk+i} = A_i$ for any integer k.



Write this system of equations in the form $c^{j+1} = Sc^j$, where c^j is the vector $\left[K^j, A_0^j, \ldots A_{n-1}^j\right]$, and S is a matrix.

b. Our task is to find the eigenvalues and eigenvectors of the matrix S. For an eigenvector x, $Sx = \lambda x$, where λ is the corresponding eigenvalue. Therefore, for an eigenvector $K^{j+1} = \lambda K^j$, and $A_i^{j+1} = \lambda A^j$. Show that unless $\lambda = 1$, the corresponding eigenvector has component $K = 0$. (Hint: sum all equations from part a.)

c. Taking the result of part b into account, the equations for $A_i^{j+1} = \lambda A_i^j$ can be rewritten as a recurrence for the sequence A_i^j. Using the standard methods for solving recurrences, find the values of λ for which it has a solution and for each λ find the solution. Keep in mind the condition $A_n^j = A_0^j$. For standard methods of solving recurrences, see, for example, Cormen,Leiserson, Rives, *Introduction to Algorithms*, or Graham, Knuth, and Patashnik, *Concrete Mathematics*.