Homework #1: Splines
Due Date: October 14

Problems marked with a $*$ are extra credit.

1. Show that any quartic (degree 4) curve $\mathbf{p}(t) = \mathbf{a}_4 t^4 + \mathbf{a}_3 t^3 + \mathbf{a}_2 t^2 + \mathbf{a}_1 t + \mathbf{a}_0$, where $\mathbf{a}_4, \mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_0$ are points in three-dimensional space, can be written as a quartic Bezier curve, and find expressions for the Bezier control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$.

2. **Degree elevation.** One way to increase the number of degrees of freedom of a curve is to raise its degree. For example, a quadratic Bezier curve is completely defined by 3 points, while a cubic Bezier curve requires 4 control points. If we have a quadratic curve we may want to get more control over its shape without chaging it initially. We can do this by regarding it as a cubic curve with the coefficient of t^3 equal to zero. To do this, we need to find 4 Bezier control points $\mathbf{b}_0^c, \mathbf{b}_1^c, \mathbf{b}_2^c, \mathbf{b}_3^c$, such that the resulting cubic curve coincides with the quadratic Bezier curve with control points $\mathbf{b}_0^q, \mathbf{b}_1^q, \mathbf{b}_2^q$. Find the expressions for $\mathbf{b}_0^c, \mathbf{b}_1^c, \mathbf{b}_2^c, \mathbf{b}_3^c$ in terms of $\mathbf{b}_0^q, \mathbf{b}_1^q, \mathbf{b}_2^q$.

3. **Linear precision.** Show that if all control points of a Bezier curve of arbitrary degree greater than zero are on the same line, the curve itself is a straight line interpolating the control points. Explain why this also holds for a B-spline curve.

4. **Linear invarinace.** We have proved that Bezier curves have translation invariance, that is, if a Bezier curve of degree n $\mathbf{p}(t)$ is defined by 2D control points $\mathbf{b}_0, \ldots \mathbf{b}_n$, then the Bezier curve defined by translated control points $\mathbf{b}_0 + \mathbf{d}, \ldots \mathbf{b}_n + \mathbf{d}$, where d is the displacement vector, is $\mathbf{p}(t) + \mathbf{d}$. Show that the curves are also linearly invariant, that is, for any 2 by 2 matrix A, the curve defined by the control points $A\mathbf{b}_0, \ldots A\mathbf{b}_n$ is $A\mathbf{p}(t)$.

5. Find the explicit expressions for the degree 4 B-spline basis functions on a uniformly spaced knot sequence with step 1.

6. Derive expressions for the first and second derivatives of a cubic B-spline curve defined on the knot sequence $r_1, r_2, \ldots r_n \ldots$. The expressions should use only the deBoor control points, knots, and the parameter value t.

7. Prove translation invariance for the tensor-product B-spline surfaces.

8. a. Find the coordinates of the deBoor control points for a NURBS representation of an ellipse $x^2/a^2 + y^2/b^2 = 1$. b*. Find the control points for NURBS surface representing an ellipsoid.

9. * Derive a necessary condition for a cubic B-spline curve to be geometrically non-C^1.