# Mesh parameterization, Distortion measures, Harmonic maps

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In this lecture, we discuss mesh parameterizations. Meshes have to be parameterized for a variety of reasons, most commonly for texture mapping. The goal is to find a mapping between the plane and a predefined mesh. Such mappings give rise to distortions. We also discuss what types of distortions may occur, how to measure them, quantitatively and how to minimize them.

## Parameterizations

Given a mesh, M, in $\mathbb{R}^3$, we want to find a map between M and a region in $\mathbb{R}^2$, which satisfies certain conditions, namely:

1. **Linearity:** On each mesh triangle, the mapping should be linear and continuous.

2. **One-to-one Mapping:** This is desirable whenever we need to compute the inverse of the parameterization. Also, if we get multiple mesh points mapping to the same plane point and use the mapping for texturing, we will get a part of the texture replicated on the mesh several times.

3. **Minimize Local Distortion** Locally, we want the map to be close to a rigid transformation in the following sense.

   We may consider our mapping to be a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ If the function is at least $C^1$, it can be approximated locally by a linear function. This local linear approximation is given by a 3 by 2 matrix $\mathbf{A}$. $\mathbf{A}$ can be thought of as mapping a region from $\mathbb{R}^2$ to the mesh tangent plane, which is in $\mathbb{R}^3$. If introduce coordinates in the tangent plane, than the transformation from 3d coordinates to the coordinates in the plane is specified by a 3 by 2 matrix $\mathbf{P}$. The composition of the two transformations maps the parametric plane to the tangent plane and is specified by a 2 by 2 matrix $\mathbf{B}$:

   $$\mathbf{B} = \mathbf{PA},$$

   If $\mathbf{B}$ is a rigid transformation (rotation or reflection), then there is no local distortion to worry about. However, if not, then we can use invariants of $\mathbf{B}$ to measure various types of local distortion.

There are two main types of distortion with which we are concerned:

1. **uniform scaling** – distortion of size;

2. **shear and non-uniform scaling** – distortion of shape.

A convenient way of thinking about the distortion introduced by linear transforms is to consider the image of a unit square and a unit circle.

The unit square is mapped by the transform $\mathbf{B}$ to the parallelogram with sides equal to the vectors specified by the columns of $\mathbf{B}$. It is easy to check that the area of this parallelogram is $|\det \mathbf{B}|$.

The points on a unit circle satisfy $\|v\|^2 = 1$, i.e. $v^T v = 1$. If we apply the transformation $B$, than the image of a point $v$ is a point $w = Bv$, or $v = B^{-1}w$. Therefore, the points $w$ of the image of the unit circle satisfy $w^T(BB^T)^{-1}w = 1$. This equation defines an ellipse.

The major axes of the ellipse are given by the singular values of $BB^T$. more specifically the matrix can be written in the following form

$$\mathbf{BB} = \mathbf{R}^T \begin{pmatrix} \pm\lambda_1^2 & 0 \\ 0 & \pm\lambda_2^2 \end{pmatrix} \mathbf{R}$$

for some choice of signs, and for some $\mathbf{R}$ satisfying $\mathbf{R}\mathbf{R}^T = I$. $\lambda_1$ and $\lambda_2$ are called singular values of $B$. We can see that $\det \mathbf{B}^T\mathbf{B}$ is, one the one hand $(detB)^2$ and, on the other hand, $\lambda_1^2\lambda_2^2$. we conclude that the measure of size distortion is $\lambda_1\lambda_2$; to make it symmetric with respect to $\lambda_1$ and $\lambda_2$, we can use $\frac{1}{2}(\lambda_1/\lambda_2 + \lambda_2/\lambda_1)$.

The ratio $\lambda_1/\lambda_2$ defines the aspect ratio of image of the unit circle. If this ratio is 1, then the image is also a circle, possibly of a different radius; in this case there is no shape distortion. So we can use the ratio of the singular values as the measure of shape distortion. Next we express the measures of distortion directly in terms of derivatives of $f$.

The matrix $A$ of the local linear approximation of $f$ has the partial derivatives of $f$ as columns.

$$\mathbf{A} = \begin{pmatrix} f_u & f_v \end{pmatrix},$$

where $f_u$ and $f_v$ are both vectors in $\mathbb{R}^3$. If $e_1$ and $e_2$ are the unit basis vectors it can be seen that $f_u = \mathbf{A}e_1$ and $f_v = \mathbf{A}e_2$. Now, by considering the unit square, with its lower left corner at the origin, it can also be seen that

$$|f_u \times f_v|$$

is the area of the transformed parallelogram, which was defined by $e_1$ and $e_2$; this gives us an expression that relates the distortion of size directly to $f$.

We also note that if $f_u \perp f_v$ (equivalently $f_u \cdot f_v = 0$) and $|f_u| = |f_v|$, then there is no shape distortion. We may regard $f_u \cdot f_v$ as a measure of shear and $f_u^2 - f_v^2$ as a measure of non-uniformity of scale in the $u$ and $v$ parametric directions. Both of these measures are quantitative indicators of the degree of shape distortion, but not as geometrically natural as $\lambda_1/\lambda_2$ because they depend on the choice of coordinates in the parametric plane. It turns out that the invariant measure $\lambda_1/\lambda_2 + \lambda_2/\lambda_1$ can be computed as $\frac{1}{2}(f_u^2 + f_v^2)/|f_u \times f_v|$ (the denominator is the trace of $A^T A = B^T B$, i.e. $\lambda_1^2 + \lambda_2^2$).

**Dirichlet energy.** A simple and important observation one can make is that

$$|f_u \times f_v| \leq |f_u|\,|f_v| \leq \frac{1}{2}\left(f_u^2 + f_v^2\right),$$

Note that both equalities hold of and only if and only if there is no shape distortion, and the first quantity is the measure of size distortion.

Typically we need a measure of the total distortion introduced by a map, rather than for the local distortion at each point. The most common way to define this is to integrate over the domain. Note that this is not necessarily the most appropriate approach for a specific application.

Integrating the inequalities above we get

$$\int |f_u \times f_v|\,dudv \leq \int |f_u| * |f_v|\,dudv \leq \int \frac{1}{2}\left(f_u^2 + f_v^2\right)dudv$$

The last term is known as the Dirichlet energy; the advantage of using this energy is that it has a very simple and easy to minimize form. The map that minimizes this energy is called a harmonic map. Observe that the first term is the area of the surface, and is therefore constant, if the surface does not change; if the minimal possible value of the Dirichlet energy is attained then there is no shape distortion anywhere and the resulting map is called *conformal*. If we only fix the shape of the boundary of a parametric domain, but not the map from the boundary to the surface, a conformal parameterization always exists for smooth surfaces and smooth domain boundaries. If we fix the map on the boundary of the domain, full conformality may be impossible, but minimizing the Dirichlet energy will move us closer to a conformal map, i.e. will reduce the distortion of shape. One can minimize the integral of the distortion of shape directly, but this is a highly nonlinear functional, so it is easier to deal with Dirichlet energy.

# Shape Distortion Minimizing Parametrization

Let us now consider the problem of constructing a parameterization for a given mesh. This means that for each vertex of the mesh $p_i \in \mathbb{R}^3$ we would like to find coordinates in the plane $q_i = (u_i, v_i)$ so that a measure of distortion is minimized. We focus on a map minimizing the shape distortion by minimizing the Dirichlet energy. We observe that Dirichlet energy can be defined for piecewise linear maps, in particular for piecewise linear map the plane to the mesh: we simply define the energy as the sum of the energies of the linear maps between triangles.

So far we have defined the Dirichlet energy for the map from the plane to the surface. However, for computational purposes as we will see it is best to use the energy for the inverse map from the surface (in our case, mesh) to the plane.

While a general definition would require figuring out how to take partial derivatives on a surface, when we consider our specific problem, this is easy, as we just map triangles to triangles, and can simply introduce coordinates in the plane of the mesh triangle in $\mathbb{R}^3$. The intuition behind the energy remains the same: minimizing it makes the map closer to conformal; in particular, if $f$ minimizing the Dirichlet energy is conformal, the function minimizing the inverse is also conformal.

We consider two triangles; the first of which is in $\mathbb{R}^2$ (i.e. our parameter space), the second which is in $\mathbb{R}^3$ (i.e. our 'surface' space) then we have the following formula:

$$E_D = \sum_{i=1}^{3} cot\alpha_i |\ q_i - q_j\ |^2$$

where $\alpha_i$ represent the angles of the triangle of mesh in $\mathbb{R}^3$, and $|\ \vec{a_i}\ |^2$ is the squared length of the corresponding (i.e. opposite) side in the parametric domain.

We omit the derivation: it is a straightforward differentiation of a linear map and can be found in Pinkall and Polthier's paper.

The total energy is given by

$$\sum_i \sum_{p_j isa neighbor of p_i} (\cot\alpha_{ij} + \cot\beta_{ij}) |\ q_i - q_j\ |^2,$$

Note that this is a quadratic functional in $q_i$. To minimize it, all we need to do is to solve a linear system of equations obtained by differentiating the above energy with respect to $q_i$ and adding boundary conditions. Note that remarkably the equations that we get by differentiating the functional above are exactly the equations for the discrete approximation to the mean curvature, except $p_i$ are replaced with $q_i$.

The mapping we get in this way is a p.w. linear harmonic map, which, unlike a smooth harmonic map, is not necessarily one-to-one.

**Floater Parameterization.** An alternative to this is Floater parameterization. We observe that the general form of the system defining the parameterization based on Dirichlet energy is

$$q_i = \sum_{p_j isa neighbor of p_i} \lambda_{ij} q_j$$

where the $\lambda_{ij}$'s are factors depending on $p_i$. One can consider a general class of parameterizations which are defined by equations of this type, allowing $\lambda_{ij}$ to be chosen in arbitrary manner. Floater parametrizations are parameterizations of this type for which $\lambda_{ij} > 0$ and the boundary of the parametric domain is convex. Floater parameterizations are guaranteed to be one-to-one; at the same time, these parametrizations do not minimize any natural energy.