# Curvature and Geodesics, Discrete Laplacian and related smoothing methods

Lecture #08: Nov 4,2002
Lecturer: Prof. Denis Zorin
Scribe: Zhang, Zhongshan

Mesh smoothing is a method to remove noise or small scale features from large meshes, while still preserving the basic overall shape and important features of the original model. Most sophisticated smoothing methods rely on ideas from differential geometry. In this lecture we discuss some basic geometric concepts and introduce some simple smoothing algorithms; the paper presented at this lecture describes more sophisticated algorithms which make use of discrete approximations of differential geometric quantities.

## 1 Curvature

In 2D, curvature at a given point $p$ on curve is defined as the inverse of the radius of the osculating circle at $p$. The osculating circle can be found as follows: for any two points $p_1$ and $p_2$ near $p$ compute the unique circle passing through $p_1$, $p_2$, $p$ (if these points are collinear, than the circle has infinite radius). The osculating circle is the circle that we get in the limit if $p_1$ and $p_2$ are moved toward $p$.

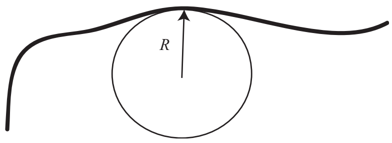



Figure 1: 2D Curvature.

$$\kappa = 1/R$$

In 3D, we have the following definitions:
A *normal curve* is the intersection of a surface with a plane containing the normal $n$. For a given direction $d$ in the tangent plane there is a unique normal curve, obtained by intersecting the plane spanned by $n$ and $d$ with the plane.

The curvature of a normal curve is called *sectional curvature.*

The *principal curvatures* $(\kappa_1, \kappa_2)$ are the maximal and minimal sectional curvatures. The *principal curvature directions* are the directions in the tangent plane for which the maximum and minimum are attained. These directions are perpendicular to each other.

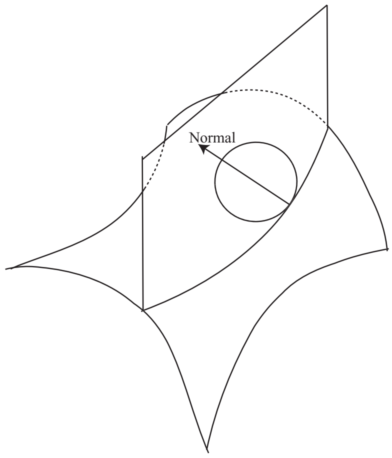



Figure 2: Curvature in 3D

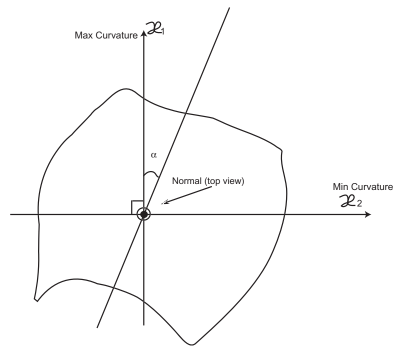



Figure 3: Curvature in 3D, top view

The sectional curvature corresponding to the direction in the tangent plane forming an angle $\alpha$ with the first principal curvature direction is given by

$$\kappa = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$$

*Gaussian curvature*is also called intrinsic curvature, and is defined as

$$\kappa_G = \kappa_1 \kappa_2$$

If the surface is isometrically deformed (i.e. the distances between points on the surface along the surface do not change) the Gaussian curvature is preserved.

*Mean Curvature* is defined by

$$\kappa_H = \frac{\kappa_1 + \kappa_2}{2}$$

Given a surface $r(u, v) : R^2 \rightarrow R^3$, let $r_u = \frac{\partial r}{\partial u}, r_v = \frac{\partial r}{\partial v}$, and the unit normal

$n = \frac{r_u \times r_v}{|r_u \times r_v|}$. Also define

$$E = (r_u \cdot r_u), F = (r_u \cdot r_v), G = (r_v \cdot r_v)$$

$$L = (n \cdot r_{uu}), M = (n \cdot r_{uv}), N = (n \cdot r_{vv})$$

Then the formula for the Gaussian curvature is

$$\kappa_G = \frac{L \cdot N - M^2}{E \cdot G - F^2}$$

and for mean curvature it is

$$\kappa_H = \frac{L \cdot G - 2F \cdot M + E \cdot N}{2(E \cdot G - F^2)}$$

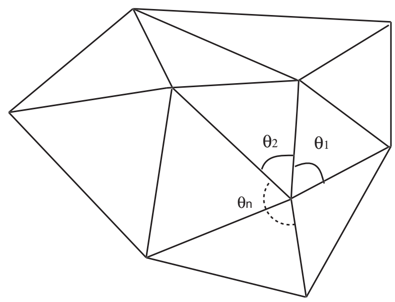



Figure 4: Angles used to compute approximation of Guassian curvature on meshes.

*Integral curvature* over an area is the integral of the curvature over that area of the surface. On meshes, one can approximate the integral Gaussian curvature over an area by

$$\sum_{i=1}^{n} \theta_i - 2\pi$$

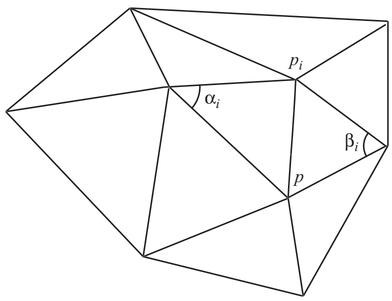
this quantity is also known as *angle excess.* It is zero if the sum of the angles at a vertex is exactly the same as it is for a planar mesh. Note that this quantity is nondimensional, which coincides with what we expect for he integral of the Gaussian curvature (measured in (length unit)$^{-2}$) over an area.

To get an estimate of the actual curvature one needs to divide by a measure of area associated with the vertex.

There is also a useful formula approximating integral mean curvature at a vertex:

$$\frac{1}{2}\sum_{i=1}^{n}(\cot \alpha_i + \cot \beta_i)\|p - p_i\|$$

where $p$ is the position of the vertex, $p_i$ are positions of surrounding vertices, $\alpha_i$ and $\beta_i$ are the two angles opposite to the edge $[p, p_i]$ on different sides of the edge.




Angles used in discrete mean curvature formula.

Figure 5: fig:mean-curv

# 2 Geodesic Curves

A *geodesic curve* is the locally shortest curve, which means for any point $x$ on curve $\gamma$, there is a neighborhood $N_\epsilon$, such that for $x_1x_2 \in (\gamma \cap N_\epsilon)$, $\gamma$ is the shortest curve in $N_\epsilon$ connecting $x_1$ and $x_2$. The geodesic on surface is locally a normal curve.

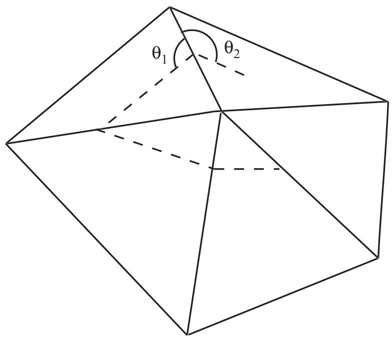



Figure 6: Geodesic Line on a mesh

Geodesics exist on meshes. A geodesic curve on a plane is a straight line; therefore on each face of the mesh, a mesh geodesic is a line segment. TO find out how the direction changes from one face to another, we simply rotate one of the faces until they are coplanar. In the plane the geodesics continues in a straight line. So the condition for continuation of a geodesic is $\theta 1 + \theta 2 = \pi$.

# 3 Mesh Smoothing

## 3.1 Laplacian Smoothing

*Laplacian smoothing* is done by simply averaging neighboring vertices iteratively. One step of the process is described by the following formula:

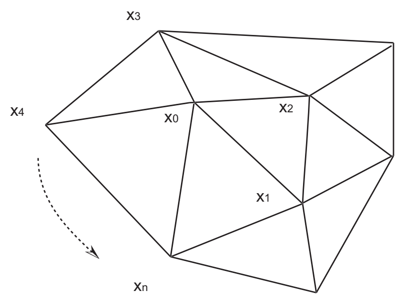



Figure 7: Laplacian Smoothing

$$x_0^{new} = x_0^{old} + \lambda \sum (\frac{x_i^{old} - x_0^{old}}{n}), 0 < \lambda < 1$$

Laplacian smoothing eliminates high-frequency noise quickly, but requires many iterations for lower frequencies. Another important problem is that Laplacian smoothing can cause shrinking. Furthermore, it causes tangential vertex motion: even if all vertices are in the plane, i.e. the surface is perfectly smooth, the vertices will be moved in the plane.

## 3.2 Taubin's Smoothing

The idea of Taubin's approach is based on the analogy with Fourier analysis for one and two dimensional regularly sampled signals.

Recall that any function $f(x)$ can be written as a combination of sin and cos functions. For simplicity we assume that the function is symmetric and only cos functions are used:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega) \cos(\omega x) d\omega$$

where $\hat{f}(\omega)$ is called the spectrum. Taubin's observation was that if we regard the basis functions of the spectrum as eigenfunctions of a continuous Laplacian (i.e. in one dimension $\partial^2/\partial x^2$, we can also try to construct a similar spectral representation using eigenvectors of the discrete Laplacian.

We can rewrite the Laplacian smoothing equation in the form

$$x_{new} = x_{old} + \lambda K x_{old} = (I + \lambda K) x_{old}$$

where $K$ is a matrix; this is the discrete Laplacian matrix.

Let $u_1 \ldots u_N$ be the eigenvectors of $K$, where $N$ is the total number of vertices in mesh.

Then we can expand the vector of initial vertex position as

$$x^{init} = \sum_{i=1}^{N} a_i u_i$$

The eigenvalues of the eigenvectors correspond to frequencies in the continuous case. Smoothing corresponds to eliminating terms in the sum for which eigenvalues $\lambda_i$ are big.

Taubin has shown that applying two steps of Laplacian smoothing, one with positive and one with negative coefficient repeatedly has the desired effect.

The coefficients $\mu$ and $\lambda$ for sequential steps should be replaced by

$$\frac{1}{\lambda} + \frac{1}{\mu} = CutOffFrequency$$

where $\lambda$ is positive and $\mu$ is negative.

The number of steps is chosen empirically as it is the case for Laplacian smoothing.