Smoothness of Stationary Subdivision on Irregular Meshes

Denis Zorin
Courant Institute of Mathematical Sciences
New York University*

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Abstract

We derive necessary and sufficient conditions for tangent plane and C^k-continuity of stationary subdivision schemes near extraordinary vertices. Our criteria generalize most previously known conditions. We introduce a new approach to analysis of subdivision surfaces based on the idea of the *universal surface*. Any subdivision surface can be locally represented as a projection of the universal surface, which is uniquely defined by the subdivision scheme. This approach provides us with a more intuitive geometric understanding of subdivision near extraordinary vertices.

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1 Introduction

The goal of this work is to establish conditions that are both necessary and sufficient for tangent plane continuity and C^k-continuity of subdivision surfaces at extraordinary vertices. We propose an approach to the analysis of subdivision near extraordinary vertices based on the idea of *universal surfaces*. Any subdivision surface in $\mathbf{R}^3$ can be regarded as a projection of a unique (up to an affine transformation) universal surface in a higher-dimensional space. The analysis of smoothness of a subdivision scheme can be reduced to the analysis of smoothness of the corresponding universal surface. The advantage of this approach is that it exposes the geometric origin of the smoothness conditions and allows us to decrease the complexity of the derivations without sacrificing the generality.

This paper extends a number of previous results, most importantly, the results of Reif [23] and Prautzsch [19]. We extend the previous work in several ways. We concentrate on conditions that are necessary and sufficient simultaneously, thus providing descriptions of certain classes of subdivision schemes. We eliminate or make explicit most assumptions that were implied by other authors. We do not assume that the subdivision scheme is reduced to spline subdivision on regular complexes; we do not make any assumptions on the eigenstructure of the subdivision matrix as it is done, for example, in [23]. The conditions for C^k-continuity proposed in [19] are a special case of our conditions. It is important to note, however, that we do assume that the subdivision scheme produces C^k-continuous limit functions on regular grids, and concentrate on the extraordinary vertices. Powerful methods exist for analysis of smoothness on regular grids (see, for example, [4, 5, 8, 9]).

Motivation. Our primary motivation is to build a more general theory of smoothness of subdivision surfaces near extraordinary vertices. From the practical point of view, more general conditions are also desirable for a number of reasons.

Analysis of schemes. In certain cases, the common assumptions of existing conditions are not satisfied for certain schemes; for example, the Butterfly scheme and midedge subdivision scheme of Peters and Reif [18] for certain valences have nontrivial Jordan blocks corresponding to dominant eigenvalues; piecewise-subdivision schemes similar to the scheme of Hoppe *et al.* [13] may have a characteristic map with identically vanishing Jacobian.

*This work was done at the Computer Science Departments of the California Institute of Technology and Stanford University, in 1996-98

Stability and parametric families of schemes. Most commonly used schemes (Catmull-Clark, Doo-Sabin, Loop, Butterfly) have subdivision matrices with largest eigenvalues $1, \lambda$, with $1 > \lambda$ and λ having geometric multiplicity 2. As a consequence, arbitrarily small perturbations of coefficients of the scheme result in matrices that do not satisfy common assumptions on subdivision matrices. If we consider parametric families of schemes, it is also likely that for certain values of parameters we obtain schemes with derogatory subdivision matrices.

Construction of new schemes. Understanding of the structure of the classes of tangent plane continuous and C^k-continuous schemes would help to make better choices when constructing new schemes.

Previous work. First conditions for tangent plane continuity of subdivision surfaces were discussed by Doo and Sabin [6] and Ball and Storry [2]. Halstead *et al.* [11] also discuss analysis of subdivision surfaces. Sufficient conditions for C^1-continuity were proposed by Reif [23] (our results are briefly compared to Reif's results in Sections 3.3 and 4). As we have already mentioned, sufficient conditions and partial necessary conditions for C^k-continuity are given in Prautzsch [19]. More recently, specific schemes were analyzed by Habib and Warren [10], Schweitzer [24] and Peters and Reif [17]. Several ideas of this paper originate in the work of Warren [25]. The idea of the universal surface was suggested to us by Tom Duchamp.

Overview. We consider subdivision surfaces defined on simplicial complexes. Similar to the regular setting, a surface $f : |K| \to \mathbf{R}^3$ defined on a simplicial complex K can be decomposed into a sum of basis functions:

$$f[p](y) = \sum_v p(v)\varphi_v(y) \tag{1.1}$$

where $p(v)$ are the control points in $\mathbf{R}^3$.

For schemes with finite support we need only a finite number of basis functions to represent the surface near an extraordinary vertex. Let ψ be the vector of basis functions φ_v that contribute to f on the neighborhood U_1 of an extraordinary vertex, consisting of the triangles of the complex adjacent to the vertex. Equation (1.1) can be written in vector form $f[p] = (p, \psi)$, One of the crucial ideas of this paper becomes apparent from this vector equation: we regard the surfaces $f[p]$ generated by subdivision as projections of a higher-dimensional surface defined by $\psi : U_1 \to \mathbf{R}^p$, which we call the *universal surface*.

If S is the subdivision matrix, that is, the control points p^{j+1} near the extraordinary vertex can be computed from control point p^j using $p^{j+1} = Sp^j$, then the subdivision surface is invariant under the action of the adjoint of the subdivision matrix on $\mathbf{R}^p$:

$$\psi(y/2) = S^T\psi(y) \tag{1.2}$$

We show that almost all possible surfaces generated by subdivision are tangent plane continuous at extraordinary vertices of a fixed valence, if and only if the universal surface is tangent plane continuous; same is true for C^k-continuity.

To find conditions on the universal surface ψ and subdivision matrix S that are necessary and sufficient for tangent plane continuity, we use the following observation: the tangent planes to the universal surface can be characterized by the wedge products of tangent vectors $w(y) = \partial_1\psi(y) \wedge \partial_2\psi(y)$; the subdivision matrix defines a linear operator on $\mathbf{R}^p$; this operator can be extended to $\Lambda^2(\mathbf{R}^p)$, the space of 2-vectors. Let ΛS be the matrix of the extended operator. The structure of the *tangent subdivision matrix* ΛS is completely determined by the structure of S. Using this matrix, we can write a scaling relation for the normals:

$$w(y/2) = 4\Lambda S^T w(y)$$

It follows from this relation that the sequence of 2-vectors defining the tangent planes at points $y, y/2, y/4 \ldots$ is $w(y), \Lambda S^T w(y), (\Lambda S^T)^2 w(y), \ldots$. This fact indicates that the question of tangent plane continuity can be reduced to a general question of convergence of directions of vectors $A^s v$ to a common direction, $s = 0 \ldots$ for a matrix A and some initial vector v. Once conditions on ΛS are established, we can reduce them to the conditions on S.

Conditions for tangent plane continuity form the foundation of our results. C^1-continuity conditions are immediately obtained from tangent plane continuity conditions using a general geometric fact (Proposition 1.2).

Once C^1 continuity is established, the universal surface can be thought of as a function over the tangent plane with an isolated singularity; the scaling relation imposes further constraints. The scaling relation (1.2) suggests that the components of ψ in a suitably chosen basis are homogeneous functions; if the surface is reparameterized over the tangent plane, the components of the new parameterization are *quasihomogeneous functions*. The conditions for C^k-continuity for this type of functions are well-known in singularity theory [1, 14] for arbitrary number of variables; we re-derive these conditions for the specific case of two variables in a form more convenient for subdivision surfaces, and also consider a possible non-quasihomogeneous case.

Assumptions. In all our derivations we assume a single (although quite strong) property of the subdivision scheme in the regular setting: the subdivision scheme should produce at least C^1-continuous (and for the results on C^k-continuity, C^k-continuous) limit functions. We do not need the scheme to be stable or even stationary, as long as it is known that it produces sufficiently smooth basis functions away from extraordinary vertices, and the universal surface satisfies scaling relations. This approach has the advantage of identifying the most general properties following from the scaling relation; however, for most practical schemes it is likely to be possible to make somewhat stronger statements using other properties, for example, stability or the convex hull property for schemes with nonnegative coefficients.

Main results. This paper contains four main results. The first result is a general necessary and sufficient condition for tangent plane continuity (Theorem 3.2). In addition to C^1-continuity on regular complexes, the only assumption that we make is Condition A (Section 1.5), which is a mild nondegeneracy requirement.

With stronger nondegeneracy assumptions, which hold for most schemes, we are able to obtain simpler necessary and sufficient conditions for tangent plane continuity (Corollary 3.3 and equivalent Theorem 3.5). These results provide most insight into the properties of subdivision near extraordinary points and are central to this paper.

Without assuming Condition A, we derive sufficient conditions (Theorem 3.6), that extend the conditions of Reif [22, 23].

Finally, Theorem 4.1 gives a general necessary and sufficient condition for C^k-continuity. In an important special case when the parametric map coincides with the characteristic map (Section 3.3) we derive explicit necessary and sufficient conditions for C^k-continuity (Theorem 4.2).

It is important to note that the conditions presented in this paper have primarily theoretical value; if one desires to verify smoothness of a specific scheme, more specific conditions have to be derived. A general automatic method for verifying C^1-continuity based on the results presented in this paper is described in [27]. Necessary conditions presented here can be used to guide the construction of subdivision schemes and evaluate existing ones. In [28], we have used an extension of our results to the surfaces with boundary to detect and fix some important shortcomings of such commonly used subdivision schemes as Catmull-Clark [3] and Loop [15].

Structure of the paper. The paper is structured as follows: Section 1 gives a brief formal description of subdivision on complexes. In Section 2 we reduce the analysis of subdivision schemes to the analysis of universal surfaces.

In Sections 3.1 and 3.2 we derive criteria for tangent plane continuity. Section 3.3 describes a sufficient condition for tangent plane continuity.

In Section 4 we state the criteria for C^k-continuity.

Section 6 contains proofs of several facts used to derive the main theorems of the paper.

Several proofs are presented in a very brief form; some proofs are omitted. Complete exposition can be found in [26].

Notation[1]

$[\cdot]_+$	normalization of a vector
Φ	characteristic map, 3.3
φ	smooth parameterization of the universal surface $\psi \circ \kappa^{-1}$, 1.5
$\varphi_v(y)$	basis function at the vertex v, 1.1
κ	transformation $U_1 \to \mathbf{R}^2$, 1.5

[1]The numbers after each item in the list indicate in which section the symbol is defined

$\Lambda^2(\mathbf{R}^p)$	space of wedge products of vectors from $\mathbf{R}^p$, 1.3
λ_i	eigenvalues of the subdivision matrix, 1.2
Ψ	parametric map, 3.2
ψ	universal surface $U_1 \to \mathbf{R}^p$, 1.2
b^i_{jr}	complex generalized eigenvectors of the subdivision matrix, 1.2
c^i_{jr}	real generalized eigenvectors of the subdivision matrix, 1.2
$\mathrm{Ctrl}(V)$	control set of a set of vertices V, 1.4
D_ψ	directional set, 3.1
e^i_{jr}	basis vectors of the basis for the subdivision matrix S^T, 1.2
$f[p]$	limit function $f : \|K\| \to B$ generated by subdivision from $p \in \mathcal{P}(V, B)$, 1.1
f^i_{jr}	complex eigenbasis functions, 1.2
g^i_{jr}	real eigenbasis functions, 1.2
h^i_{jr}	basis vectors of the basis for the subdivision matrix S, 1.2
$J[\cdot]$	the Jacobian of a mapping $\mathbf{R}^2 \to \mathbf{R}^2$, 1.5
J^i_j	a cyclic subspace of a matrix with eigenvalue λ_i, 1.2
K	complex, 1.1
K^j	the result of subdividing K j times, also denoted $D^j(K)$, 1.1
$N_m(v, K)$	m-neighborhood of the vertex v, 1.1
n^i_j	order of the cyclic subspace J^i_j, 1.2
$\mathbf{P}(p, q)$	quasihomogeneous polynomials, 4
$\bar{\mathbf{P}}(p, q)$	complex quasihomogeneous polynomials, 4
$\mathcal{P}(V, B)$	the space of functions on the set of vertices of V with values in B, 1.1
$\mathrm{Proj}(v, W)$	projection of a vector onto the subspace W spanned by a set of basis vectors, 3.1
$\mathcal{R}$	regular complex, 1.1
$\mathcal{R}_k$	k-regular complex, 1.1
S	subdivision matrix or subdivision scheme, 1.1, 1.2
$S[K]$	subdivision operator $\mathcal{P}(V) \to \mathcal{P}(V^1)$, 1.1
ΛS	tangent subdivision matrix, 3.1
$\|K\|$	complex regarded as a topological space, 1.1
U_m	topological m-neighborhood of the central vertex of a k-regular complex, 1.2
$w(y)$	$p(p-1)/2$-dimensional vector $\partial_1\psi \wedge \partial_2\psi$, 1.5

sectionSubdivision Schemes: Formal Description and Concepts of Smoothness

1.1 Subdivision on Complexes

This section is a brief formal description of general subdivision.

Simplicial complexes. Subdivision surfaces are naturally defined as functions on two-dimensional simplicial complexes. Recall that a simplicial complex K is a set of vertices, edges and triangles in $\mathbf{R}^N$, such that for any triangle all its edges are in K, and for any edge its vertices are in K. We assume that there are no isolated vertices or edges. $|K|$ denotes the union of triangles of the complex regarded as a subset of $\mathbf{R}^N$ with induced metric. We say that two complexes K_1 and K_2 are *isomorphic* if there is a homeomorphism between $|K_1|$ and $|K_2|$ that maps vertices to vertices, edges to edges and triangles to triangles.

A *subcomplex* of a complex K is a subset of K that is a complex. A 1-neighborhood $N_1(v, K)$ of a vertex v in a complex K is the subcomplex formed by all triangles that have v as a vertex. The m-neighborhood of a vertex v is defined recursively as a union of all 1-neighborhoods of vertices in the $(m-1)$-neighborhood of v. We omit K in the notation for neighborhoods when it is clear what complex we refer to.

Recall that a *link* of a vertex is the set of edges of $N_1(v, K)$ that do not contain v. We consider only complexes with all vertices having links that are connected simple polygonal lines, open or closed. If the link of a vertex is an open polygonal line, this vertex is a boundary vertex, otherwise it is an internal vertex.

Most of our constructions use two special types of complexes — *k-regular complexes* $\mathcal{R}_k$ and the *regular complex* $\mathcal{R}$. Each complex is simply a triangulation of the plane consisting of identical triangles. In the regular complex each vertex has exactly 6 neighbors. In a k-regular complex all vertices have 6 neighbors, except one vertex C, which has k neighbors. We call C the central vertex of a k-regular complex and identify it with zero in the plane.

Subdivision of simplicial complexes. We can construct a new complex $D(K)$ from a complex K by subdivision, adding a new vertex for each edge of the complex and replacing each old triangle with four new triangles. Let m_{vw} be the midpoint of the edge (v, w); if (u, v, w) is a triangle of K, then (u, m_{uv}, m_{uw}), (v, m_{vw}, m_{uv}), (w, m_{uw}, m_{vw}) and (m_{uv}, m_{vw}, m_{uw}) are triangles of $D(K)$. Note that k-regular complexes are self-similar, that is, $D(\mathcal{R}_k)$ and $\mathcal{R}_k$ are isomorphic.

We use notation K^j for the j times subdivided complex $D^j(K)$ and V^j for the set of vertices of K^j. Note that the sets of vertices are nested: $V^0 \subset V^1 \subset \dots$. We call the elements of the union $\cup_{i=0}^{\infty} V^i$ the *dyadic points* of K.

Subdivision schemes. Next, we attach values to the vertices of the complex; in other words, we consider the space of functions $V \to B$, where B is a vector space over $\mathbf{R}$. The range B is typically $\mathbf{R}^l$ or $\mathbf{C}^l$ for some l. We denote this space $\mathcal{P}(V, B)$, or $\mathcal{P}(V)$, if the choice of B is not important.

A *subdivision scheme* for any function $p^j(v)$ on vertices V^j of the complex K^j computes a function $p^{j+1}(v)$ on the vertices of the subdivided complex $D(K^j) = K^{j+1}$. More formally, a subdivision scheme is a collection of operators $S[K]$ defined for every complex K, mapping $\mathcal{P}(V)$ to $\mathcal{P}(V^1)$. We consider only subdivision schemes that are linear, that is, the operators $S[K]$ are linear functions on $\mathcal{P}(K)$. In this case the subdivision operators are defined by equations

$$p^1(v) = \sum_{w \in V} a_{vw} p^0(w)$$

for all $v \in V^1$. The coefficients a_{vw} may depend on K. We restrict our attention to subdivision schemes which are finitely supported, locally invariant with respect to a set of isomorphisms of complexes and affinely invariant.

A subdivision scheme is *finitely supported* if there is an integer M such that $a_{vw} \neq 0$ only if $w \in N_M(v, K)$ for any complex K (note that the neighborhood is taken in the complex K^{j+1}). We call the minimal possible M the *support size* of the scheme.

We assume our schemes to be *locally defined* and *invariant with respect to a set of isomorphisms* G. Together these two requirements can be defined as follows: there is a constant L such that if for two complexes K_1 and K_2 and two vertices $v_1 \in V_1$ and $v_2 \in V_2$ there is an isomorphism $\rho : N_L(v_1, K_1) \to N_L(v_2, K_2)$, $\rho \in G$ such that $\rho(v_1) = v_2$, then $a_{v_1 w} = a_{v_2 \rho(w)}$. In most cases, the *localization size* $L = M$.

We assume that the set G contains isomorphisms of 1-neighborhoods of any vertex of any complex with a subcomplex of a k-regular complex possibly with boundary. In addition, if it contains an isomorphism $\rho : K_1 \to K_2$, it also contains the induced isomorphism of $D(K_1) \to D(K_2)$, as well as the restrictions of ρ to subcomplexes of K_1.

For most common schemes, the set G coincides with all possible isomorphisms of complexes. An example of a nontrivial set G is the set of isomorphisms of tagged complexes: we can tag some edges of the complex, and propagate the tags to the edges of the subdivided complex. We can allow only isomorphisms that map tagged edges to tagged edges. Analysis of quadrilateral-based schemes, such as Catmull-Clark and Doo-Sabin, can be reduced to analysis of subdivision schemes on complexes introducing auxiliary vertices into complexes and tagging certain edges. Schemes on tagged complexes also can be used to create surfaces with creases. The requirement that we impose on the set G guarantees that the surfaces generated by subdivision on arbitrary complexes are locally identical to the surfaces generated by subdivision on a k-regular complex, possibly with boundary (see below.)

The final requirement that we impose on subdivision schemes is *affine invariance*: if T is a linear transformation $B \to B$, then for any $v \quad Tp^{j+1}(v) = \sum a_{vw} T p^j(v)$. This is equivalent to requiring that all coefficients a_{vw} for a fixed v sum up to 1.

Limit functions. For each vertex $v \in \cup_{j=0}^{\infty} V^j$ there is a sequence of values $p^i(v)$, $p^{i+1}(v)$, $\dots$ where i is the minimal number such that V^i contains v.

Definition 1.1. *A subdivision scheme is called convergent on a complex K with vertex set V, if for any function $p \in \mathcal{P}(V, B)$ there is a continuous function f defined on $|K|$ with values in B, such that*

$$\lim_{j \to \infty} \sup_{v \in V^j} \left\| p^j(v) - f(v) \right\|_2 \to 0$$

The function f is called the limit function of subdivision.

Notation: $f[p]$ is the limit function generated by subdivision from the initial values $p \in \mathcal{P}(V)$.

It is easy to show that if a limit function exists, it is unique. A *subdivision surface* is the limit function of subdivision on a complex K with values in $\mathbf{R}^3$. In this case we call the initial values $p^0(v)$ the *control points* of the surface.

Similar to Theorem 2.1 of [4] we can represent any limit function of subdivision as a linear combination of *basis functions*. A basis function $\varphi_v(y) : |K| \to \mathbf{R}$ at a vertex v is obtained from the initial values $\delta_v \in \mathcal{P}(V, \mathbf{R})$, $\delta_v(v) = 1$, $\delta_v(w) = 0$ if $w \neq v$. Let p^0 be some initial values on a complex K. If subdivision converges,

$$f[p^0](y) = \sum_{v \in V^0} p^0(v)\varphi_v(y) \tag{1.3}$$

Reduction to k-regular complexes. Locally any surface generated by a subdivision scheme on an arbitrary complex can be thought of as a part of a subdivision surface defined on a k-regular complex, if the set of isomorphisms G, with respect to which the scheme is invariant, satisfies the requirements above. The reason for this can be easily understood from Figure 1. More formally this can be proved by establishing isomorphisms between neighborhoods $N_L(v, K^j)$ of any vertex of K^j for sufficiently large j and neighborhoods $N_L(0, \mathcal{R}_k)$ of the central vertex of the k-regular complex or regular complex and proving that they are in G.

Note that this fact alone does not guarantee that it is sufficient to study subdivision schemes only on k-regular complexes (see Section 1.3).

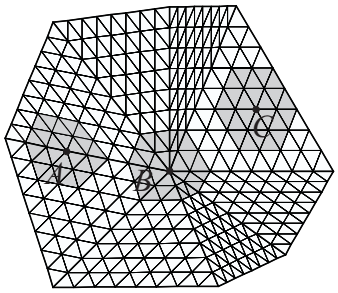

Figure 1: Neighborhoods of vertices A, B and C isomorphic to neighborhoods in regular (A and C) and k-regular complexes; $L = 2$.

If the complex has boundary, we also need to consider regular and k-regular complexes with boundaries. We concentrate on the analysis for closed surfaces, and do not consider the boundary case.

The schemes for subdivision surfaces are typically constructed from schemes that generate C^k-continuous limit functions $f[p]$ on a regular complex. We assume that this is the case, and focus on C^k-continuity near extraordinary vertices.

1.2 Subdivision Matrices

We have already observed that we have to consider primarily k-regular complexes, which are just triangulations of the plane. Consider the part of a subdivision surface $f[y]$ with $y \in U_1^j = |N_1(0, \mathcal{R}_k^j)|$, defined on the k-gon formed by triangles of the subdivided complex $\mathcal{R}_k^j$ adjacent to the central vertex. It is straightforward to show that the values at all dyadic points in this k-gon can be computed given the initial values $p^j(v)$ for $v \in N_L(0, \mathcal{R}_k^j)$. In particular, the control points $p^{j+1}(v)$ for $v \in N_L(0, R_k^{j+1})$ can be computed using only control points $p^j(w)$ for $w \in N_L(0, \mathcal{R}_k^j)$. Let $\bar{p}^j$ be the vector of control points $p^j(v)$ for $v \in N_L(0, \mathcal{R}_k^j)$. Let $p + 1$ be the number of vertices in $N_L(0, \mathcal{R}_k)$.

As the subdivision operators are linear, $\bar{p}^{j+1}$ can be computed from $\bar{p}^j$ using a $(p + 1) \times (p + 1)$ matrix S^j:

$$\bar{p}^{j+1} = S^j \bar{p}^j$$

If for some m and for all $j > m$, $S^j = S^m = S$, we say that the subdivision scheme is *stationary on the k-regular complex*, or simply stationary, and call S the *subdivision matrix* of the scheme. Note that our definition in the case $k = 6$ is weaker than the standard definition of stationary schemes on regular complexes [4].

As we will see, eigenvalues and eigenvectors of the matrix have fundamental importance for smoothness of subdivision.

Eigenbasis functions. let $\lambda_0, \lambda_i, \ldots \lambda_J$ be different eigenvalues of the subdivision matrix. The following lemma, proved in [26], is similar to a lemma of Reif [23]:

Lemma 1.1. *If a subdivision scheme converges on the regular complex, it is necessary and sufficient for convergence on a k-regular complex that the subdivision matrix S has eigenvalue 1 with a single cyclic subspace of size 1 and all other eigenvalues have magnitude less than 1.*

Let $\lambda_0 = 1$. For any λ_i let J^i_j, $j = 1 \ldots$ be the complex cyclic subspaces corresponding to this eigenvalue.

Let n^i_j be the *orders* of these cyclic subspaces; the order of a cyclic subspace is equal to its dimension minus one.

Let b^i_{jr}, $r = 0 \ldots n^i_j$ be the complex generalized eigenvectors corresponding to the cyclic subspace J^i_j. The vectors b^i_{jr} satisfy

$$S b^i_{jr} = \lambda_i b^i_{jr} + b^i_{j\,r-1} \quad \text{if } r > 0, \quad S b^i_{j0} = \lambda_i b^i_{j0} \tag{1.4}$$

We use the following rules for enumerating the cyclic subspaces of S:

- All eigenvalues are enumerated in the order of nonincreasing magnitude.
- If the magnitudes of eigenvalues are equal, they are enumerated in the order of nonincreasing order of the largest cyclic subspace.
- If the eigenvalues have equal magnitudes, and equal orders of highest-order cyclic subspace, real eigenvalues have smaller numbers than complex; the real positive eigenvalue if there is one, has number less than real negative; two complex-conjugate eigenvalues have sequential numbers; the order of complex-conjugate pairs of eigenvalues is insignificant for our purposes.
- For each eigenvalue the cyclic subspaces are enumerated in nonincreasing order, i.e., $n^i_1 \geq n^i_2 \geq n^i_3 \geq \ldots$.

The complex *eigenbasis functions* are the limit functions defined by $f^i_{jr} = f[b^i_{jr}] : U_1 \to \mathbf{C}$. It immediately follows from (1.3) that any subdivision surface $f[p] : U_1 \to \mathbf{R}^3$ can be represented as

$$f[p](y) = \sum_{i,j,r} \beta^i_{jr} f^i_{jr}(y) \tag{1.5}$$

where $\beta^i_{jr} \in \mathbf{C}^3$, and if $b^i_{jr} = \overline{b^k_{lt}}$, $\beta^i_{jr} = \overline{\beta^k_{lt}}$, where the bar denotes complex conjugation.

One can show using the definition of limit functions of subdivision and (1.4) that the eigenbasis functions satisfy the following set of *scaling relations*:

$$f^i_{jr}(y/2) = \lambda_i f^i_{jr}(y) + f^i_{j\,r-1}(y) \quad \text{if } r > 0, \quad f^i_{j0}(y/2) = \lambda_i f^i_{j0}(y) \tag{1.6}$$

Real eigenbasis functions. As we consider real surfaces, it is often convenient to use real Jordan normal form of the matrix rather than the complex Jordan normal form. For any pair of the complex-conjugate eigenvalues λ_i, λ_k, we can choose the complex cyclic subspaces in such a way that they can be arranged into pairs J^i_j, J^k_j, and $b^i_{jr} = \overline{b^k_{jr}}$ for all j and r. Then we can introduce a single real subspace for each pair, with the basis c^i_{jr}, c^k_{jr}, $r = 0 \ldots n^i_j$, where $c^i_{jr} = \Re b^i_{jr}$, and $c^k_{jr} = \Im b^i_{jr}$. We call such subspaces *Jordan subspaces.* Then we can introduce real eigenbasis functions $g^i_{jr}(y) = f^i_{jr}(y)$ for real λ_i, and $g^i_{jr}(y) = \Re f^i_{jr}(y)$, $g^k_{jr}(y) = \Im f^i_{jr}(y)$ for a pair of complex-conjugate eigenvalues (λ_i, λ_k). For a Jordan subspace corresponding to pairs of complex eigenvalues the order is the same as the order of one of the pair of cyclic subspaces corresponding to it. We follow the same rules for enumerating Jordan spaces, with one alteration: instead of two sequences of cyclic subspaces corresponding to a pair of complex Jordan eigenvalues we have a single sequence of Jordan subspaces.

Similar to (1.5) we can write for any surface generated by subdivision on U_1:

$$f[p](y) = \sum_{i,j,r} \alpha^i_{jr} g^i_{jr}(y) \tag{1.7}$$

Now all coefficients α^i_{jr} are real. Eigenbasis functions corresponding to the eigenvalue 0 have no effect on tangent plane continuity or C^k-continuity of the surface at zero. From now on we assume that $\lambda_i \neq 0$ for all i.

We can assume that the coordinate system in $\mathbf{R}^3$ is always chosen in such a way that the single component of $f[p]$ corresponding to eigenvalue 1 is zero. This allows us to reduce the number of terms in (1.7) to p.

Universal map. The decomposition (1.7) can be written in vector form. Let h^i_{jr} be an orthonormal basis of $\mathbf{R}^p$. Let ψ be $\sum_{i,j,r} g^i_{jr} h^i_{jr}$; this is a map $U_1 \to \mathbf{R}^p$. Let $\alpha^1, \alpha^2, \alpha^3 \in \mathbf{R}^p$ be the vectors composed of components of coefficients α^i_{jr} from (1.7) (each of these coefficients is a vector in $\mathbf{R}^3$). Then (1.7) can be rewritten as

$$f[p](y) = \big((\psi, \alpha^1),\, (\psi, \alpha^2),\, (\psi, \alpha^3)\big) \tag{1.8}$$

This equation indicates that all surfaces generated by a subdivision scheme on U_1 can be viewed as projections of a single surface in $\mathbf{R}^p$. We call ψ the *universal map,* and the surface specified by ψ the *universal surface.* The universal map plays an important role in our constructions.

In the chosen basis the matrix S is in the real Jordan normal form. Note that by definition of S for any $a \in \mathbf{R}^p$

$$(a, \psi(y/2)) = (Sa, \psi(y))$$

Using the well-known formula for inner products $(Su, v) = (u, S^T v)$, we get

$$(x, \psi(y/2)) = (x, S^T \psi(y)), \quad \text{for any } x$$

This means that the scaling relations can be jointly written as

$$\psi(y/2) = S^T \psi(y) \tag{1.9}$$

Although S^T is not in Jordan normal form, a simple permutation of the vectors of the basis reduces S^T to Jordan normal form; specifically, for the Jordan subspace of a real eigenvalue λ_i of order n_i, introduce a new basis $e^i_{jr} = h^i_{j\, n_j - r}$, that is, simply reverse the order of basis vectors.

1.3 Tangent Plane Continuity and C^1-continuity

We are going to use the following definition of C^1-continuity of a surface:

Definition 1.2. *Consider a surface (M, f) where M is a topological space, and f is a map $f : M \to \mathbf{R}^p$. For any $x \in M$ and a neighborhood $U_x \subset M$, let h be a parameterization of $f(U_x)$ over a disk D in the plane, that is, a map $h : D \to f(U_x) \subset \mathbf{R}^p$. The surface (M, f) is called* C^1**-continuous** *if for any x there is a neighborhood U_x and a parameterization h which is regular, that is, C^1-continuous and with Jacobi matrix of maximal rank (2). If the parameterization h can be chosen to be C^k, then the surface is* C^k**-continuous.**

Surfaces satisfying this definition are two-dimensional manifolds immersed in $\mathbf{R}^3$. Note that the parameterization h and the map f are not necessarily related. For subdivision surfaces defined on a complex K, $M = |K|$ and f is the limit function of subdivision.

Tangent plane continuity. It turns out to be useful to split the task of establishing C^1-continuity of a subdivision surface at extraordinary vertices into two steps: first, check the existence of a tangent plane, then determine if the projection into the tangent plane is injective (see Proposition 1.2 below).

In $\mathbf{R}^3$ planes are conveniently characterized by their normals. Our plan is to reduce analysis of subdivision to analysis of the universal surface. In order to achieve this, we need a characterization of tangent planes of 2-dimensional surfaces in $\mathbf{R}^p$. Instead of the normals, we can use *2-vectors*, which are elements of $\Lambda^2(\mathbf{R}^p)$. Any plane in $\mathbf{R}^p$ spanned by vectors v_1, v_2 corresponds to a one-dimensional subspace in $\Lambda^2(\mathbf{R}^p)$ spanned by $v_1 \wedge v_2$. It is more convenient for our purposes to consider *oriented tangent planes*, which correspond to directions in $\Lambda^2(\mathbf{R}^p)$ (sets of vectors of the form $kv_1 \wedge v_2$, where $k > 0$). There is a unique 2-vector of length 1 for each direction. We denote this 2-vector $[v_1 \wedge v_2]_+$. For $p = 3$, this 2-vector can be identified with the unit normal to the plane.

Now we can define tangent plane continuity in $\mathbf{R}^p$:

Definition 1.3. *Let D be the unit disk in the plane. Suppose a surface (M, f) in a neighborhood of a point $x \in M$ is parameterized by $h : D \to \mathbf{R}^p$, which is regular everywhere except 0, and $h(0) = f(x)$. Let $\pi(y) = [\partial_1 h \wedge \partial_2 h]_+$. where $\partial_1 h$ and $\partial_2 h$ are derivatives with respect to the coordinates in the plane of the disk D. The surface is* **tangent plane continuous** *at x if the limit $\lim_{y\to 0} \pi(y)$ exists.*

It is possible to characterize tangent planes in such a way that no orientation is specified; then the limit plane might exist, even if the limit of consistently oriented normals does not. One can show that such surfaces are not C^1-continuous; as we regard tangent plane continuity as an intermediate stage on the way to C^1-continuity, we choose a somewhat stronger definition including orientation.

The following Proposition shows the relation between tangent plane continuity and C^1-continuity:

Proposition 1.2. *Suppose a surface is tangent plane continuous at zero. Let τ be the limit at zero of the oriented tangent planes, and let $P_\tau : \mathbf{R}^p \to \mathbf{R}^2$ be the projection of $\mathbf{R}^p$ onto the plane τ. Then the surface is C^1-continuous if and only if there is a neighborhood D of zero, such that P_τ restricted to $f(D)$ is injective.*

This proposition is proved in Section 6.1.

1.4 C^1-continuous Subdivision Schemes

It would be natural to say that a subdivision scheme is C^1-continuous if all surfaces generated by a scheme are C^1-continuous. However, this requirement is too restrictive: in general, it is impossible to construct schemes of this type; even for spline surfaces we can find configurations of control points that lead to non-C^1-continuous surfaces. We adopt a weaker notion of C^1-continuity of a scheme. Recall that the collections of control values for a given complex can be regarded as elements of a linear space $\mathcal{P}(V)$. As we consider only local schemes, it is sufficient to consider only finite complexes. For such complexes, the spaces $\mathcal{P}(V)$ are finite-dimensional, and we can define a distance on $\mathcal{P}(V)$ identifying it with a Euclidean space. We consider a subdivision scheme C^1-continuous on a complex K if it generates C^1 continuous surfaces for all initial values $p \in \mathcal{P}(V)$ excluding a nowhere dense subset of $\mathcal{P}(V)$.

This approach introduces a new problem. For a vertex v, let $\mathrm{Ctrl}(N_1(v, K))$ be the set of vertices $w \in V$ such that the values of the limit function $f[p]$ on $|N_1(v, K)|$ depend only on the vertices from $\mathrm{Ctrl}(N_1(v, K))$. Recall that we reduce the analysis of subdivision on arbitrary complexes to analysis on k-regular complexes using an isomorphism ρ between $N_L(v, K^j)$ and $N_L(0, \mathcal{R}_k)$ for some j. Clearly, the values $p^j(v)$ on $N_L(v, K^j)$ can be computed from the values $p^0(v)$ on $\mathrm{Ctrl}(v)$. By linearity of subdivision, there is a matrix (not necessarily square) A such that $p^j(v) = Ap^0(v)$. If the rank of A is less than $p + 1$, then the dimension of the space of $p^j(v)$ on $N_L(v, K^j)$ is less than the maximal dimension $p + 1$ and it can be identified with a proper subspace $\tilde{\mathcal{P}}$ of the space $\mathcal{P}_k$ of functions on the vertices of $N_L(0, \mathcal{R}_k)$, rather than with the whole space. The simplest example of such complex is a tetrahedron: the dimension of $\tilde{\mathcal{P}}$ cannot be more than 4, but even for Loop scheme $p = 9$ for a vertex of valence 3. It might happen that the subspace $\tilde{\mathcal{P}}$ is contained inside the nowhere dense subset of $\mathcal{P}_k$ for which subdivision generates surfaces that are not C^1-continuous. We call complexes for which this occurs *constraining*. It is difficult to characterize constraining complexes for arbitrary schemes. We simplify our task by excluding such complexes.

This leads us to the following definition:

Definition 1.4. *A subdivision scheme is C^1-continuous on a complex K if it generates C^1-continuous surfaces for any choice of control points on K, except a nowhere dense set of configurations. A subdivision scheme is C^1-continuous, if it is C^1-continuous for any non-constraining complex.*

Tangent plane continuity of a subdivision scheme is defined in a similar way. This definition allows us to consider only subdivision on k-regular complexes. If a subdivision scheme is C^1-continuous according to our criteria, additional analysis is needed to identify constraining complexes.

1.5 Singular Parameterizations

To be able to analyze tangent plane continuity of subdivision at extraordinary vertices of valence k, we need a parameterization p used in the definition, which is regular on a neighborhood of zero in the k-regular complex, excluding zero itself. We cannot use the map ψ directly because the partial derivatives of ψ do not exist on the boundaries between triangles of the k-gon U_1 unless $k = 6$. A regular away from zero parameterization can be obtained using a construction similar, but not identical, to the complex-analytic structure on complexes described by Duchamp and others [7].

Consider the 1-neighborhood U_1 of the extraordinary vertex of the k-regular complex. The surface $f : U_1 \to \mathbf{R}^3$ defined by subdivision is C^1-continuous by assumption in the interior of the triangles of U_1 and may be not C^1-continuous on the boundaries between triangles. However, we can map any pair of adjacent triangles to two adjacent triangles of the regular complex using a piecewise-linear mapping h; Then $f \circ h^{-1}$ has to be C^1 on the interior of the quadrilateral formed by the two triangles of the regular complex.

Note that any deformation of the two triangles of U_1 that agrees with h in the limit near the boundary between the two triangles, and is at least C^1 in the interior, can be used instead of h. We describe a mapping κ defined on the whole neighborhood U_1 that agrees with mappings h for each pair of adjacent triangles of U_1.

To define the map κ, we identify the plane with the complex plane. Suppose the vertices of the k-gon U_1 are π/k, $3\pi/k$, $\ldots$ $(2k-1)\pi/k$. Let h_k be the linear scaling $(u, v) \to \left(\frac{\cos \pi/6}{\cos \pi/k} u, \frac{\sin \pi/6}{\sin \pi/k} v\right)$. Let $\chi(z)$ be the map $\chi_k(z) = z^{6/k}$. The image of the equilateral triangle with vertices 0, $e^{\pi/6}$, $e^{-\pi/6}$ is contained in the triangle T_0, with two of the edges adjacent to 0 mapping to the edges of T_0.

Then on the triangle T_m with vertices 0, $(2m-1)\pi/k$, $(2m+1)\pi/k$ the map κ can be defined as

$$\kappa(z) = e^{-2im\pi/k} \left(\chi_k \left(h_k(e^{2im\pi/k} z)\right)\right) \tag{1.10}$$

The structure of the mapping κ is shown in Figure 2

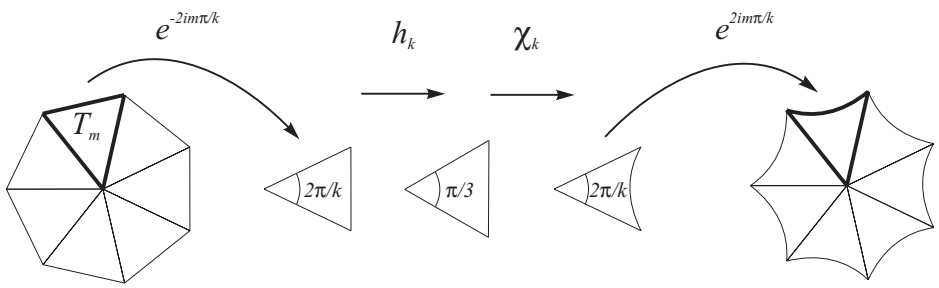

Figure 2: Construction of the singular C^1-continuous parameterization κ.

Then for any surface $f : U_1 \to \mathbf{R}^3$ generated by subdivision, the parameterization $f \circ \kappa^{-1}$ is C^1-continuous everywhere except 0; same is true for the parameterization $\varphi = \psi \circ \kappa^{-1}$ of the universal surface. This parameterization need not be regular.

To be able to reduce analysis of subdivision to analysis of the universal map, we assume that $\varphi = \psi \circ \kappa^{-1}$ is regular, which is equivalent to the following condition:

Condition A. For any $y \in \kappa(U_1)$

$$\partial_1 \varphi(y) \wedge \partial_2 \varphi(y) \neq 0 \quad \text{for all } y \in \kappa(U_1),\, y \neq 0$$

If Condition A is violated, then there is a point in U_1 such that any surface generated by the subdivision scheme would have a singularity there. Moreover, one can see from scaling relations for wedge products of tangents (3.1), that there will be a singularity arbitrarily close to zero. In this work we consider mostly schemes satisfying Condition A. However, we do not necessarily assume *a priori* that Condition A is satisfied; in certain cases, weaker assumptions are sufficient; in other cases, Condition A is implied by other assumtions. When we assume Condition A, we mention this explicitly.

Normals of subdivision surfaces. There is a simple formula relating the Jacobian of a mapping $U_1 \to \mathbf{R}^2$ generated by subdivision to the wedge product $\partial_1\psi \wedge \partial_2\psi$. The Jacobian of a mapping $f[x^1, x^2] = ((x^1, \psi), (x^2, \psi))$ is

$$J[f[x^1, x^2]] = (x^1 \wedge x^2, \partial_1\psi \wedge \partial_2\psi) \tag{1.11}$$

Note that the partial derivatives of ψ are defined only on the interior of the triangles of U_1; on the boundaries only one-sided derivatives exist, excluding zero. Therefore, only one-sided limits of the Jacobian are defined on the boundaries between triangles. However, one can show using the expression $J[\psi] = J[\varphi]J[\kappa^{-1}]$ that these one-sided limis actually coincide, and the Jacobian is continuous on U_1 away from zero. Equation (1.11) is valid on the boundaries too, if one-sided derivatives of ψ are used.

Equation (1.11) is useful for relating the normals of subdivision surfaces in $\mathbf{R}^3$ to the normals of the universal surface. For a surface $F[x^1, x^2, x^3] : U_1 \to \mathbf{R}^3$ a normal at any point except zero can be written as

$$N(y) = [(x^2 \wedge x^3, w(y)), (x^3 \wedge x^1, w(y)), (x^1 \wedge x^2, w(y))], \quad y \in \kappa(U) \tag{1.12}$$

where $w(y) = \partial_1\psi(y) \wedge \partial_2\psi(y)$. Note that Condition A implies that $w(y) \neq 0$ for all y. Therefore, for any choice of x^1, x^2, x^3, such that at least 2 vectors are independent, the vector above is not zero and the unit normal can be obtained by normalizing the vector above.

2 Reduction to the Analysis of the Universal Surfaces

Our goal is to relate tangent plane continuity and C^k-continuity of the universal surface in $\mathbf{R}^p$ and tangent plane continuity of the subdivision scheme. The following theorem holds under our assumptions:

Theorem 2.1. *For a subdivision scheme satisfying Condition A to be tangent plane continuous on a k-regular complex, it is necessary and sufficient that the universal surface be tangent plane continuous; for the subdivision scheme to be C^k-continuous, it is necessary and sufficient that the universal surface be C^k-continuous.*

Proof. Sufficiency is straightforward.
Necessity: tangent plane continuity. Suppose the universal surface is not tangent plane continuous at zero, that is, the limit $\lim_{y\to 0}[w(y)]_+$ does not exist.

Note that $[w(y)]_+ \in S^{p(p-1)/2-1}$, the unit sphere in $\Lambda^2(\mathbf{R}^p)$. As the sphere is compact, there are two sequences y_s^1, y_s^2, $s \in \mathbf{N}$ such that $\lim_{s\to\infty}[w(y_s^1)]_+ = u_1$, $\lim_{s\to\infty}[w(y_s^2)]_+ = u_2$, and $u_1 \neq u_2$. As the set of all decomposable elements in $\Lambda^2(\mathbf{R}^p)$ is closed, $u_1 = u_1^1 \wedge u_1^2$ and $u_2 = u_2^1 \wedge u_2^2$ for some $u_1^1,\ u_1^2,\ u_2^1,\ u_2^2 \in \mathbf{R}^p$. As both u_1 and u_2 are unit vectors and are not equal, at least 3 out of 4 vectors $u_1^1,\ u_1^2,\ u_2^1,\ u_2^2$ are linearly independent, or $u_1 = -u_2$.

For the purposes of this proof it is convenient to fix a basis such that $u_j^i, i, j = 1, 2$ are vectors of the basis (if some u_j^i are linearly dependent, we can always modify our choices of u_j^i so that the only ones that are dependent, are equal). We assume that u_1^i, and u_2^i are independent for $i = 1, 2$. If there are three independent vectors, we assume that u_1^1 and u_2^2 are independent. Otherwise, $u_1^1 = u_2^2$ and $u_2^1 = u_1^2$.

First, assume that at least 3 vectors u_j^i are independent. For any basis $e_i, i = 1 \dots p$ in $\mathbf{R}^p$ we can construct a basis in $\Lambda^2(\mathbf{R}^p)$ out of vectors $e_i \wedge e_j$, $i < j$. For the dual basis $\tilde{e}_i$, the corresponding basis $\tilde{e}_i \wedge \tilde{e}_j$ in $\Lambda^2(\mathbf{R}^p)$ is dual to the basis $e_i \wedge e_j$. Let $\tilde{u}_j^i$ be the vectors dual to u_j^i, that is, satisfying $(\tilde{u}_j^i, u_l^k) = \delta(i-k)\delta(j-l)$, and orthogonal to other vectors of the basis.

Consider the surface $[\tilde{u}_1^1, \tilde{u}_2^1 + \tilde{u}_1^2, \tilde{u}_2^2]$. The normals to this surface are given by

$$N(y) = \left[((\tilde{u}_2^1 + \tilde{u}_1^2) \wedge \tilde{u}_2^2, [w(y)]_+), (\tilde{u}_2^2 \wedge \tilde{u}_1^1, [w(y)]_+), (\tilde{u}_1^1 \wedge (\tilde{u}_2^1 + \tilde{u}_1^2), [w(y)]_+)\right]$$

Note that the limit $\lim_{s\to\infty} N(y_s^1)$ is $[0,0,1]$ if all four vectors are independent; if $u_2^1 = u_1^2$, the limit is $[0,0,2]$. For $N(y_s^2)$ we get $[1,0,0]$ if all four vectors are independent; if $u_2^1 = u_1^2$, we get $[2,0,0]$. In either case, the sequences of unit normals $[N(y_s^1)]_+$ and $[N(y_s^2)]_+$ converge to different limits.

In the case of two independent vectors among u_i^j the argument is similar.

In both cases it is easy to see that any surface obtained from the described surfaces by small perturbation will not be tangent plane continuous. Therefore, there is a set of surfaces of measure greater than zero that are not tangent plane continuous, and the scheme is not tangent plane continuous.

Necessity, C^k-continuity. We assume that the universal surface is tangent plane continuous; for the surface to be C^1-continuous, it is necessary and sufficient for the surface to have injective projection into the tangent plane in a neighborhood of zero (Proposition 1.2). Suppose the projection of the universal surface into the tangent plane is not injective arbitrarily close to zero. As we have seen, the tangent plane is spanned by two basis vectors in $\mathbf{R}^p$. Suppose these vectors are u_1^0 and u_2^0, and ψ_1, ψ_2 are the corresponding components of the universal map $\psi : U_1 \to \mathbf{R}^p$. Let map Ψ be the map $(\psi_1, \psi_2) : U_1 \to \mathbf{R}^p$. Let τ be the tangent plane, $P_\tau : \mathbf{R}^p \to \tau$ be the projection into the tangent plane defined by $x \in \mathbf{R}^p \to ((u_1^0, x), (u_1^0, x))$. If $P_\tau|_{\psi(U_1)}$ is not injective arbitrarily close to zero, then there are two sequences of points $y_s^1, y_s^2 \in U_1$, $s = 1\ldots$, such that $\psi(y_s^1) \neq \psi(y_s^2)$ for all s, $\lim_{s\to\infty} y_s^1 = \lim_{s\to\infty} y_s^2 = 0$ and $\Psi(y_s^1) = \Psi(y_s^2)$. We can choose a component ψ_i of ψ that has different values at infinitely many pairs of points y_s^1, y_s^2. Consider a surface in $\mathbf{R}^3$ defined by (ψ_1, ψ_2, ψ_i). The tangent plane to this surface is obtained by projecting u_1^0 and u_2^0 into $\mathbf{R}^3$; this plane coincides with the plane spanned by the first two coordinate axes in in $\mathbf{R}^3$. Clearly, projection into this plane is not injective. Now consider arbitrary projection of ψ into $\mathbf{R}^3$. By a change of coordinates, we can always reduce it to the form (ψ_1, ψ_2, f) where f is a linear combination of components of ψ. If this linear combination is sufficiently close to ψ_i, the projection is not injective again. We have constructed an open set of surfaces generated by subdivision that are not C^1-continuous, and the scheme cannot be C^1-continuous.

The argument is easily extended to C^k-continuous surfaces: for the universal surface to be C^k-continuous it is necessary and sufficient that the inverse of the projection to the tangent plane is C^k-continuous. As any subdivision surface in $\mathbf{R}^3$ can be obtained by applying a linear mapping $P : \mathbf{R}^p \to \mathbf{R}^3$ to the universal surface, the projection of the surface in $\mathbf{R}^3$ into its tangent plane is obtained in the same way. We have shown that if the universal surface is C^1-continuous for almost any linear mapping P the projection into the tangent plane is injective. Then its inverse is well defined and its derivatives can be computed as linear combinations of the derivatives of the parameterization of the universal surface over its tangent plane. If the universal surface is not C^k-continuous, for almost any choice of P the subdivision surface in $\mathbf{R}^3$ is not C^k-continuous. □

3 Tangent Plane Continuity

3.1 Tangent Plane Continuity Criterion

In this section we are going to formulate a general criterion for tangent plane continuity of the universal surface. We make very few assumptions about the subdivision scheme:

- eigenbasis functions are C^1-continuous on regular complexes;
- Condition A;
- the scaling relation $\psi(y/2) = S^T\psi(y)$, $y \in U_1$.

Scaling relation holds for any scheme which is stationary on k-regular complexes. It is important to keep in mind that although eigenbasis functions for a stationary subdivision scheme necessarily satisfy scaling relations, the converse is not true, that is, not every set of functions satisfying scaling relations can be generated by subdivision. We primarily explore properties of the universal surface that do not depend on the fact that the coordinate functions of the surface were obtained by subdivision.

Action of the subdivision matrix on tangents. As we are interested in the behavior of the tangent planes to the universal surface, rather than using the scaling relation for the surface, it is convenient to formulate a scaling relation for the elements of $\Lambda^2(\mathbf{R}^p)$.

We obtain the action of S on $\Lambda^2(\mathbf{R}^p)$ by setting

$$\Lambda S(u_1 \wedge u_2) = Su_1 \wedge Su_2$$

This defines the action on decomposable elements. It is easy to see that ΛS is linear and can be extended by linearity to the whole space $\Lambda^2(\mathbf{R}^p)$. We call the matrix of ΛS with respect to the basis $h^i_{jr} \wedge h^k_{lt}$ the *tangent subdivision matrix*.

Recall that the scaling relations can be written as $\psi(y/2) = S^T\psi(y)$. Differentiating and taking wedge products,

$$w(y/2) = 4\Lambda S^T w(y) \tag{3.1}$$

where $w(y) = \partial_1\psi(y) \wedge \partial_2\psi(y)$. Again, although only one-sided partial derivatives exist on the boundaries of triangles of U_1, the wedge product does not depend on the chosen triangle; thus, $w(y)$ is well-defined on U_1 away from zero.

If the 2-vectors $w(y)$, $y \in U_1$ span the whole space $\Lambda^2(\mathbf{R}^p)$, as we will see below, the smoothness properties of the scheme are mostly determined by the eigenstructure of ΛS. In general, however, this is not the case: it is possible that two or more functions generated by subdivision are dependent, i.e., $J\big[f[x^1], f[x^2]\big](y) = 0$ for all y. In this case the tangents to the surface are constrained to the directions perpendicular to the plane $x^1 \wedge x^2$. Writing the Jacobian above as $(x^1 \wedge x^2, \partial_1\psi \wedge \partial_2\psi)$ we can see that the condition for dependence of two functions generated by subdivision can be written in $\Lambda^2(\mathbf{R}^p)$ as orthogonality to the space spanned by vectors $w(y)$, $y \in U_1$. The set of all directions of $w(y)$ is the p-dimensional analog of the set of the directions of normals, i.e., the image of the Gauss map of the surface.

Definition 3.1. *The* **directional set** D_ψ *is the image of the Gauss map* $[\partial_1\psi(y) \wedge \partial_2\psi(y)]_+ : U_1 \to S^{p(p-1)/2-1}$.

The crucial property of the directional set D_ψ trivially follows from the scaling relation for tangents: *if* $v \in D_\psi$, *then* $[\Lambda S^T v]_+ \in D_\psi$.

Asymptotic behavior of vectors under iterated linear transforms. It follows from relations (3.1) that sequences of 2-vectors of the form $[(\Lambda S^T)^s u]_+$ are important for analysis of tangent plane continuity. The behavior of such sequences is best understood if we identify $\Lambda^2(\mathbf{R}^p)$ with the Euclidean space $\mathbf{R}^{p(p-1)/2}$ and regard 2-vectors just as vectors. We need to determine the conditions on a matrix A and vector $v \in \mathbf{R}^k = \mathbf{R}^{p(p-1)/2}$ that are necessary and sufficient for convergence of the sequence $[A^s v]_+$ as $s \to \infty$. The conditions for convergence of such sequences are quite general and have little to do with subdivision. Here we just state the main definitions and the condition for convergence (Lemma 3.1). The proof can be found in Section 6.2. We make only one assumption on A: no eigenvalue of A is equal to zero and all eigenvalues are less than 1. These assumptions do not lead to a loss of generality.

For each eigenvalue let V_μ be the corresponding invariant space, that is, the subspace of vectors that are annulled by $(A - \mu I)^j$ for some j. The *order* of any vector v in the invariant subspace V_μ of a matrix A is the minimal number j such that $v \in \mathrm{Ker}(A - \mu I)^{j+1}$.

If a vector $v \in V^j_\mu$ has order k, then $Av = \mu v + v'$ where v' has order $k-1$. By induction we obtain the following decomposition of $A^s v$ for $s \geq k$:

$$A^s v = \mu^s \sum_{q=0}^{k} \mu^{q-k} \begin{pmatrix} s \\ k-q \end{pmatrix} v^{(q)} \tag{3.2}$$

where $v^{(q)}$ is in V^q_μ, and $v^{(q)} \neq 0$. As $s \to \infty$, the direction of Av converges to the direction of $v^{(0)}$.

A decomposition similar to (3.2) can be written for complex eigenvalues. Let χ be the complex phase of the eigenvalue μ, let $v^{(q)}_1 = \Re v^{(q)}$, $v^{(q)}_2 = \Im v^{(q)}$. where $v^{(q)}$ are complex generalized eigenvectors of order q.

$$A^s v = |\mu|^s \sum_{q=0}^{k} |\mu|^{q-k} \begin{pmatrix} s \\ k-q \end{pmatrix} v^{(q)}_1 \cos((s+q-k)\chi) - v^{(q)}_2 \sin((s+q-k)\chi) \tag{3.3}$$

Consider an arbitrary vector v in $\mathbf{R}^k$. The vector v can be written as a linear combination of the vectors in the invariant subspaces V_μ of A:

$$v = \sum_{\mu} v_\mu \tag{3.4}$$

where $v_\mu \in V_\mu$. We also use notation $\mathrm{Proj}(v, V_\mu)$ for v_μ. This decomposition is unique. Let k_μ be the order of the vector v_μ, if $v_\mu \neq 0$.

For a given vector v, Let $M = \max\{|\mu| \Big| v_\mu \neq 0\}$, and $k_M = \max\{k_\mu \Big| v_\mu \neq 0,\ |\mu| = M\}$. Define $\mathcal{M}(v)$ as $\{\mu \Big| \mu = M, k_\mu = k_M\}$. The set $\mathcal{M}(v)$ identifies components of $A^s v$ that determine the asymptotic behavior of the direction of $A^s v$. These are the components with coefficients changing as $M^s s^{k_M}$, as $s \to \infty$ with maximal possible M and k_M.

Lemma 3.1. *For a fixed given v, if there is a complex or negative $\mu \in \mathcal{M}(v)$, $A^s v$ does not have a limit direction as $s \to \infty$. Otherwise, $\mathcal{M}(v)$ has a single positive element M and the limit direction is given by $u^0(v) = \left[v_M^{(0)}\right]_+$; The sequence $\left\|u^0(v) - [A^s v]_+\right\|$ converges to zero no slower than Cs^{-1}.*

We apply this Lemma to the tangent subdivision matrix ΛS acting on 2-vectors.

Tangent plane continuity criterion. We are ready to state a general criterion for tangent plane continuity. Recall that $[4\Lambda S^T w(y)]_+ = [\Lambda S^T w(y)]_+ = [w(y/2^n)]_+$, $s = 0 \ldots$, defines a sequence of tangent planes at points y, $y/2$ $\ldots$ in U_1. It is clearly necessary for existence of a limit tangent plane that all such sequences converge to the same limit. It turns out to be sufficient. Note that the factor 4 in (3.1) has no effect on the limit direction, therefore, we can drop it and consider sequences $(\Lambda S^T)^s w(y)$. From now on we will drop this factor.

Let V_μ be the invariant subspace of $\Lambda^2(\mathbf{R}^p)$ corresponding to the eigenvalue μ of ΛS^T. Let $u_\mu = \mathrm{Proj}(u, V_\mu)$ be the component of a 2-vector u from the invariant subspace V_μ of ΛS. For a set of 2-vectors X, $\mathrm{Proj}(X, V_\mu)$ is the set of $\mathrm{Proj}(u, V_\mu)$ for all $u \in X$.

Lemma 3.1 allows us to prove the following general condition for tangent plane continuity:

Theorem 3.2. *The universal surface and hence the corresponding subdivision scheme is tangent plane continuous at zero, if and only if there is a real positive eigenvalue M of ΛS^T and an eigenvector u^0 of M such that the following conditions hold for all u from the directional set D_ψ:*

1. *the set $\mathcal{M}(u)$ contains a single element M;*
2. *for any 2-vector $u \in D_\psi$, let $u_M = \mathrm{Proj}(u, V_M)$; then the term $u_M^{(0)}$ in decomposition* (3.2) *is au^0 for some $a \neq 0$; in other words, if the order of u_M is k, then u_M is in the preimage $(\Lambda S^T - MI)^{-k}(\mathrm{span}\langle u^0\rangle)$ and is not zero.*

Proof. Necessity. The first condition immediately follows from Lemma 3.1. By definition of $\mathcal{M}$, the projection u_μ is non-zero. In addition, the limit direction is the same for all 2-vectors; this means that in the expansion (3.2) $u^{(0)}$ is the same for all $u \in D_\psi$ up to a scaling factor. Given that $u_M^{(0)} = (S - MI)^k u_M$ for an element of order k, we obtain the second condition of the lemma.

Sufficiency. The conditions of the lemma guarantee that for any 2-vector u, $[(\Lambda S^T)^s u]_+$ converges to the same limit u^0. Lemma 3.1 gives us a uniform estimate for the convergence rate of the direction of $(\Lambda S^T)^s u$. Consider a ring R^0 in U_1 with outer radius $2r$ and inner radius r. The distance to the limit direction $\left\|[w(y)]_+ - u^0\right\|$ is bounded by some constant K on the ring R^0. Let R^j be the ring with inner radius $r/2^j$ and outer radius $r/2^{j-1}$. Then on R^j the distance to the limit direction can be estimated from above by CKj^{-1}, where C is a constant not depending on y or j, as R^j is compact. The same estimate applies to the union of rings R^s, $s = j \ldots$, that is, to a punctured neighborhood of zero. We conclude that the direction of $w(y)$ regarded as a function of y has a limit at 0. □

3.2 Tangent plane continuity of schemes with nondegenerate directional sets

The results presented in this section, while being less general than the results of the previous sections, are of primary importance both for practical purposes and for understanding the geometry of subdivision surfaces near extraordinary vertices.

A geometrically natural assumption on the directional set D_ψ is that $\mathrm{span}\langle D_\psi\rangle$ has maximal possible dimension, that is, coincides with $\Lambda^2(\mathbf{R}^p)$. This assumption holds if the universal surface is in a general position — any surface can be deformed into a general position surface by arbitrarily small perturbation. In three dimensions, this is equivalent to requiring that the surface is not a cylinder: there is no plane such that the projection of ψ into this plane is a curve. In this case, for any generalized eigenvector e of ΛS, we are guaranteed to have 2-vectors $\partial_1\varphi \wedge \partial_2\varphi \in D_\psi$ with non-zero component along e.

Corollary 3.3. *Suppose that for a subdivision scheme with a universal map ψ $\mathrm{span}\langle D_\psi\rangle = \Lambda^2(\mathbf{R}^p)$. Then the subdivision scheme is tangent plane continuous if and only if*

1. *The subdivision matrix ΛS^T has an eigenvalue of maximal magnitude M, which is positive and real, and this eigenvalue has a single cyclic subspace J_M of maximal order k_M (dominant cyclic subspace); for any eigenvalue μ, such that $|\mu| = M$, the maximal order of a cyclic subspace is less than k_M.*
2. *For any $u \in D_\psi$, $\mathrm{Proj}(u, J_M) \neq 0$.*

Proof. If $\mathrm{span}\langle D_\psi\rangle = \Lambda^2(\mathbf{R}^p)$ then for any 2-vector u, and hence for any invariant subspace $V_\mu \in \Lambda^2(\mathbf{R}^p)$, $\mathrm{Proj}(D_\psi, V_\mu) \neq 0$. Then the first condition follows from the first condition of Theorem 3.2.

If the eigenvalue M has two cyclic subspaces of maximal order k_M, there is a subspace W of V_M with all 2-vectors of order k_M of dimension at least two. The projection of D_ψ on that subspace should span a two-dimensional subspace. Therefore, we can find two 2-vectors u_1 and u_2 from D_ψ such that $u_1' = \mathrm{Proj}(u_1, W)$ and $u_2' = \mathrm{Proj}(u_2, W)$ are linearly independent. By construction of W, $\alpha_1 u_1' + \alpha_2 u_2'$ also has order k_M for any α_1 and α_2, unless both are 0. Note that the limit directions of $[(\Lambda S^T)^s u_1]_+$ and $[(\Lambda S^T)^s u_2]_+$ are $u_1^\infty = \left[(\Lambda S - MI)^{k_M} u_1\right]_+$ and $u_2^\infty = \left[(\Lambda S - MI)^{k_M} u_2\right]_+$ respectively. As $\alpha_1 u_1' + \alpha_2 u_2'$ has order k_M, $\alpha_1 u_1^\infty + \alpha_2 u_2^\infty \neq 0$ if one of α_1, α_2 is not zero. Therefore, $(\Lambda S^T)^s u_1$ and $(\Lambda S^T)^s u_2$ have different limit directions. We conclude that the cyclic subspace of maximal order must be unique.

The second condition of the corollary directly follows from Theorem 3.2. □

There are some interesting cases for which the assumptions of Corollary 3.3 are not satisfied; most notable exception are piecewise-smooth schemes of the type described by H. Hoppe and others [13]. The assumption is easy to verify for piecewise-polynomial schemes, as for such schemes Jacobians also can be expressed in polynomial bases, and the nondegeneracy assumption is reduced to checking independence of vectors of control values for the Jacobians.

The conditions on D_ψ and ΛS^T required by Corollary 3.3 are quite simple. In practice, however, it is more useful to have explicit conditions on eigenbasis functions rather than on the directional set D_ψ, and on the matrix S^T, rather than on the larger matrix ΛS^T. There are three parts of Corollary 3.3 that have to be restated: the assumption about $\mathrm{span}\langle D_\psi\rangle$, the conditions on the eigenstructure of ΛS^T and the condition on the projection of D_ψ on the dominant cyclic subspace of ΛS^T. Now we consider these parts one by one.

Linear independence of Jacobians. First, we reformulate the assumption of Corollary 3.3 in terms of eigenbasis functions. Observe that the components of the vectors $w(y) \in D_\psi$ are the Jacobians $J[g^i_{jr}, g^k_{lm}](y)$. $\mathrm{span}\langle D_\psi\rangle = \Lambda^2(\mathbf{R}^p)$, if and only if for any vector $u \in \Lambda^2(\mathbf{R}^p)$, there is y such that $(u, w(y)) = 0$; the latter inequality means that for any linear combinations of Jacobians $J[g^i_{jr}, g^k_{lm}](y)$, there is a point y such that this linear combination is not zero, that is, the set of functions $J[g^i_{jr}, g^k_{lm}](y)$ is linearly independent.

Eigenstructure of ΛS^T. To interpret the condition imposed by Corollary 3.3 on the tangent subdivision matrix, we use a Lemma relating the eigenstructure of a matrix ΛB acting on $\Lambda^2(\mathbf{R}^p)$ to the eigenstructure of the matrix B acting on $\mathbf{R}^p$. This Lemma is a general algebraic fact and is not specific to subdivision. We use the notation for eigenvalues and cyclic subspaces of B introduced in Section 1.2 for the subdivision matrix and the order of cyclic subspaces fixed there. We use an ordering of pairs (λ_i, n^i_j) corresponding to the order of cyclic subspaces: $(\lambda_i, n^i_j) > (\lambda_k, n^k_l)$, if $|\lambda_i| > |\lambda_k|$, or $|\lambda_i| = |\lambda_k|$ and $n^i_j > n^k_l$.

Let $\Pr\left(J^i_j \wedge J^k_l\right)$ be the real cyclic subspace generated by the vector $e^i_{j\,n^i_j} \wedge e^k_{l\,n^k_l}$ if $J^i_j \neq J^k_l$, and by $e^i_{j\,n^i_j} \wedge e^i_{j\,n^i_j-1}$ otherwise (we assume that $\lambda_1\lambda_k$ is real). As it is shown in Section 6.3, this is the largest cyclic subspace of $J^i_j \wedge J^k_l$.

Lemma 3.4. *Suppose the cyclic subspaces of a matrix B are numbered following the rules described in Section 1.2. The dominant cyclic subspace J_M for the matrix ΛB acting on $\Lambda^2(\mathbf{R}^p)$ exists and corresponds to a real positive eigenvalue exactly in one of the following cases.*

1. *$J_M = \Pr\left(J_1^1 \wedge J_1^1\right)$, if λ_1 real, $(\lambda_i, n_1^i) < (\lambda_1, n_1^1 - 2)$ for all $i > 1$. If λ_1 has more than one cyclic subspace, then $n_2^1 < n_1^1 - 2$.*
2. *$J_M = \Pr\left(J_1^1 \wedge J_2^1\right)$, if λ_1 real, has at least two cyclic subspaces, $n_2^1 = n_1^1$ or $n_2^1 = n_1^1 - 1$, $(\lambda_i, n_1^i) < (\lambda_1, n_2^1)$ for all $i > 1$. If λ_1 has more than two cyclic subspaces, $n_3^1 < n_2^1$.*
3. *$J_M = \Pr\left(J_1^1 \wedge J_1^2\right)$, if λ_1 and λ_2 real, of the same sign and consequently $\lambda_1 > \lambda_2$. The eigenvalue λ_1 has a single cyclic subspace of order $n_1^1 = 0$, and $n_2^2 < n_1^2$ if n_2^2 is defined, and $(\lambda_i, n_1^i) < (\lambda_2, n_1^2)$ for all $i > 2$.*
4. *$J_M = \Pr\left(J_1^1 \wedge J_2^1\right)$, if (λ_1, λ_2) are a pair of complex-conjugate eigenvalues, and for all $i > 2$ $(\lambda_i, n_1^i) < (\lambda_1, n_1^1)$. If λ_1 has more than one cyclic subspace, then $n_1^1 > n_2^1$.*

The conditions on eigenvalues are illustrated in Figure 3. The proof of the lemma is outlined in Section 6.3.

Parametric map. Suppose the universal surface is tangent plane continuous. The limit unit 2-vector u^0 is an eigenvector of ΛS^T. As it is the limit of sequences of decomposable 2-vectors and the set of decomposable 2-vectors is closed, it can be written as $u_1^0 \wedge u_2^0$, where $u_1^0, u_2^0 \in \mathbf{R}^p$. If $u_1^0 \wedge u_2^0$ is an eigenvector of a real eigenvalue u_1^0 and u_2^0 can be chosen in one of the following ways: u_1^0 and u_2^0 are both eigenvectors, u_1^0 and u_2^0 are linear combinations of a pair of complex eigenvectors, and u_1^0, u_2^0 satisfy $S^T u_1^0 = \lambda u_1^0 + u_2^0$. For a suitable choice of the basis c_{jr}^i, u^0 has one of the forms $e_{b0}^a \wedge e_{d0}^c$ (λ_a and λ_c real), or $e_{b0}^a \wedge e_{b0}^c$ (λ_a and λ_c complex-conjugate), or $e_{b0}^a \wedge e_{b1}^a$.

Definition 3.2. *Suppose the universal surface for a subdivision scheme is tangent plane continuous, and has limit tangent plane defined by $u_1^0 \wedge u_2^0$. Then we define the* **parametric map** *as*

$$((\psi, u_1^0), (\psi, u_2^0)) : U_1 \to \mathbf{R}^2$$

The second condition of Corollary 3.3 is equivalent to requiring the parametric map to have nonzero Jacobian $J(y)$ for sufficiently small y. Indeed, for any $u = w(y)$, $J(y) = (w(y), u^0)$. If $[w(y)]_+ \to u_0$ as $y \to 0$, then $J(y)$ has to be positive as $y \to 0$. Observing that if $\mathrm{Proj}(w(y), J_\mu) = 0$ then $\mathrm{Proj}(w(y/2^s), J_\mu) = \mathrm{Proj}((\Lambda S^T)^s w(y), J_\mu) = 0$, we get the converse.

Now we have all the ingredients required to restate the Corollary 3.3 in a more explicit form.

Theorem 3.5. *Suppose the set of Jacobians of all pairs of distinct eigenbasis functions for a subdivision scheme on a k-regular complex is independent. Let S be the subdivision matrix of the scheme with eigenvalues and cyclic subspaces numbered as described in Section 1.2.*

For the subdivision scheme to be tangent plane continuous, it is necessary and sufficient that the subdivision matrix satisfies the conditions of Lemma 3.4, and for a sufficiently small neighborhood of zero, the parametric map of the scheme has positive Jacobian. The parametric map $U_1 \to \mathbf{R}^2$ is given by $(f^1_{1\,n_1^1}, f^1_{1\,n_1^1-1})$ in case 1 of Lemma 3.4, $(f^1_{1\,n_1^1}, f^1_{1\,n_2^1})$ in case 2, $(f^1_{1\,n_1^1}, f^2_{1\,n_1^2})$ in case 3, and $(\Re f^1_{1\,n_1^1}, \Im f^2_{1\,n_1^1})$ in case 4.

The conditions of the theorem are illustrated in Figure 3.

The theorem is just a restatement of Corollary 3.3 in a different language.

Comparison of Corollary 3.3 and equivalent Theorem 3.5 shows the advantage of using the tangent subdivision matrix ΛS for theoretical analysis: otherwise, the geometric properties of subdivision are obscured by the apparent complexity of the conditions on eigenvalues and generalized eigenvectors.

A necessary condition. Corollary 3.3 can be used to derive a necessary condition on the subdivision matrix for tangent plane continuity: suppose that the tangent subdivision matrix has two dominant blocks J_M^1 and J_M^2 of order k_M. If there are points $y_1, y_2 \in U_1$ such that the projections of $w(y_1)$ and $w(y_2)$ on $V_M = J_M^1 \oplus J_M^2$ have maximal order, and $\mathrm{Proj}\,(w(y_1), V_M)^{(0)}$ and $\mathrm{Proj}\,(w(y_1), V_M)^{(0)}$ are linearly independent, then the scheme is not tangent plane continuous. Similarly to Theorem 3.5, this statement can be expressed as a condition on the eigenbasis functions and on the subdivision matrix; the condition on the matrix is the same as in Theorem 3.5. To prove this condition, we do not need to assume Condition A. The details can be found in [26].

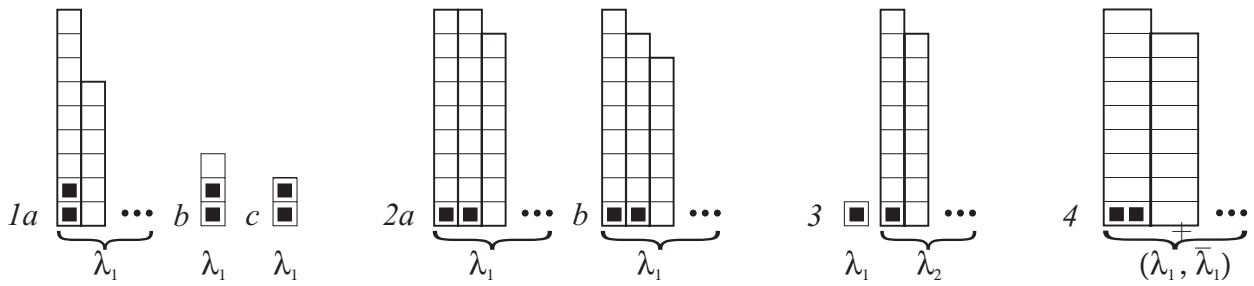

Figure 3: Conditions of Lemma 3.4 illustrated graphically. Each column corresponds to a Jordan subspace. Each cell in the columns corresponds to a generalized eigenvector (pair of generalized eigenvectors for complex eigenvalues) of matrix S^T. The generalized eigenvectors generating the parametric map are marked with black squares.

3.3 Sufficient Conditions for Tangent Plane Continuity

In the previous sections we have derived conditions for tangent plane continuity that are geometrically natural, but only in Theorem 3.3 we have made a step towards conditions that can be explicitly verified for specific subdivision schemes. Conditions which are simultaneously necessary and sufficient are important for understanding the structure of the class of tangent plane continuous subdivision schemes. However, for the purposes of verification of tangent plane continuity of specific schemes it is more useful to have simpler conditions, even if they are less natural mathematically.

In this section we derive sufficient conditions extending those originally proposed by Reif[23][2].

As a practical criterion, Theorem 3.5 suffers from two problems: first, the assumption of the theorem is unnecessarily restrictive; second, it is likely to be difficult to evaluate Jacobian of the parametric map directly. We start with introducing a new map, called the *characteristic map*, which is closely related to the parametric map; this map is more suitable for explicit evaluation. Our definition is based on the definition proposed by Reif.

Characteristic map. Suppose a surface is tangent plane continuous, and $u^0 = \lim_{y\to 0}[w(y)]_+$, Recall that for a suitable choice of the basis c^i_{jr}, u^0 has one of the forms $e^a_{b0} \wedge e^c_{d0}$ (λ_a and λ_c real), or $e^a_{b0} \wedge e^c_{b0}$ (λ_a and λ_c complex-conjugate), or $e^a_{b0} \wedge e^a_{b1}$. We consider only the first case, the other two are similar.

Note that for the parametric map at $y/2^s$ we have

$$\left(w\left(\frac{y}{2^s}\right), u^0\right) = ((\Lambda S^T)^s w(y), u^0) = (w(y), \Lambda S^s u^0) \tag{3.5}$$

Although ΛS in the basis of wedge products $e^i_{jr} \wedge e^k_{lt}$ does not have Jordan normal form, with proper choice of ordering it still has block-diagonal form, with each block corresponding to a Jordan subspace. It is easy to show that $u^0 = e^a_{b0} \wedge e^c_{d0}$ has order $n^a_b + n^c_d = k_M$ with respect to the matrix ΛS. Therefore, as we can see from (3.2) and (3.3), asymptotically $\Lambda S^s u^0$ behaves as

$$M^s \binom{s}{k_M} e^a_{bn_b} \wedge e^c_{dn_d} = M^s \binom{s}{k_M} h^a_{b0} \wedge h^c_{d0}$$

As we have observed, the 2-vector $u^{k_M} = e^a_{bn_b} \wedge e^c_{dn_d}$ is an eigenvector of ΛS. Suppose for all $y \in U_1$, the Jacobian $(w(y), u^{k_M})$ is not zero. Then for sufficiently large s, the Jacobian of the parametric map is arbitrarily well approximated by $M^s\binom{s}{k_M}(w(y), u^{k_M})$. If the Jacobian $(w(y), u^{k_M})$ is positive, this guarantees that the parametric map has positive Jacobian sufficiently close to zero. One can observe that it is also necessary for this Jacobian to be nonnegative, otherwise the parametric map will be negative arbitrarily close to zero. These considerations lead us to the following definition:

Definition 3.3. *The* **characteristic map** $\Phi : U_1 \to \mathbf{R}^2$ *is defined for a pair of cyclic subspaces J^a_b, J^c_d of the subdivision matrix as (f^a_{b0}, f^a_{b1}) if $J^a_b = J^c_d$, λ_a is real, (f^a_{b0}, f^c_{d0}) if $J^a_b \neq J^c_d$, λ_a, λ_c are real, and $(\Re f^a_{b0}, \Im f^a_{b0})$ if $\lambda_a = \bar{\lambda_c}$, $b = d$.*

[2]Reif's conditions guarantee C^1-continuity, not just tangent plane continuity; however, as we will see in Section 4 the difference between conditions for tangent plane continuity and C^1-continuity is small.

Three types of characteristic maps are shown in Figure 4.

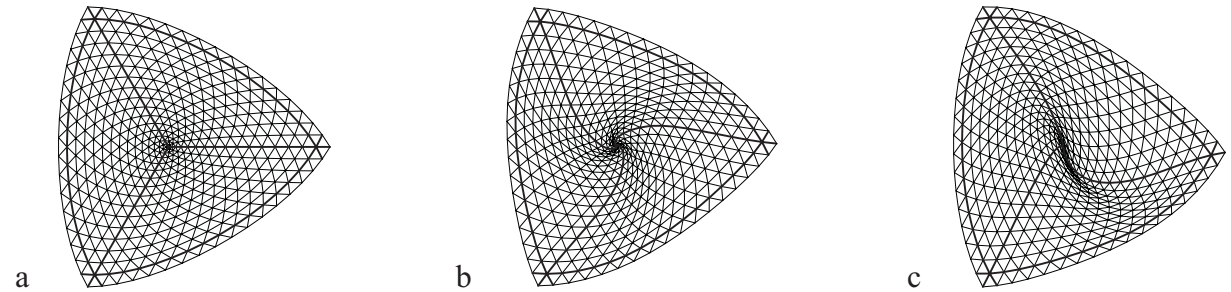

Figure 4: Three types of characteristic maps: control points after 4 subdivision steps are shown. a. Two real eigenvalues. b. A pair of complex-conjugate eigenvalues. c. Single eigenvalue with Jordan block of size 2.

Although a characteristic map is defined for many pairs of cyclic subspaces, only the map corresponding to the pair of cyclic subspace of the parametric map is of interest. The characteristic map has a remarkable property, which makes it particularly useful for proving tangent plane continuity and C^k-continuity of subdivision schemes:

The characteristic map Φ for any pair of cyclic subspaces has self-similar Jacobian:

$$J[\Phi](y/2) = J[\Phi](y)$$

This property can be easily proved using the scaling relation. Therefore, it is sufficient to verify that the characteristic map is regular on a suitably chosen annular compact set. Reif's original characteristic map is defined on such set. In our context it is more natural to consider the map defined on the whole neighborhood U_1.

Note that if the parametric map corresponds to a pair of distinct cyclic subspaces of order 0, to a cyclic subspace of a pair of complex-conjugate eigenvalues of order 0, or a single cyclic subspace of a real eigenvalue of order 1, the characteristic map coincides with the parametric map.

Sufficient condition. Now we are ready to formulate the sufficient condition. The idea of the condition is to ensure that the parametric map corresponds to a given pair of cyclic subspaces of S and then to require the corresponding characteristic map to have positive Jacobian.

Suppose that for a given pair of cyclic subspaces J^a_b, and J^c_d the characteristic map Φ has non-zero Jacobian everywhere. This guarantees that the projection of the any 2-vector in D_ψ on $\Pr(J^a_b \wedge J^c_d)$ has maximal possible order k_M, where $M = \lambda_a\lambda_c$ and $k_M = n^a_b + n^c_d$ if $J^a_b \neq J^c_d$ and $2n^a_b - 2$ otherwise.

By Theorem 3.2, it is sufficient for tangent plane continuity to ensure for any $u \in D_\psi$ that if $|\lambda_i\lambda_k| > M$, then $\mathrm{Proj}(u, J^i_j \wedge J^k_l) = 0$ and if $|\lambda_i\lambda_k| = M$, then the order of $\mathrm{Proj}(u, J^i_j \wedge J^k_l)$ is less than k_M for $(J^i_j, J^k_l) \neq (J^a_b, J^c_d)$.

The first part of this requirement is also necessary and is equivalent to $J[f^i_{jr}, f^k_{lt}] = 0$ if $|\lambda_i\lambda_k| > M$. For the second part it is sufficient to have $\mathrm{Proj}(u, e^i_{jr} \wedge e^k_{lt}) = 0$ for which the order of $e^i_{jr} \wedge e^k_{lt}$ is no less than k_M, i.e., $r + t \geq k_M$ if $(i, j) \neq (k, l)$, or $r + t - 1 \geq k_M$ if $r \neq t$ and $(i, j) = (k, l)$. However, this is not necessary: a linear combination of vectors of order k_M or higher may have order less than k_M; projections of 2-vectors from D_ψ can be such linear combinations.

Our observations lead to the following condition:

Theorem 3.6. *For a subdivision scheme to be tangent plane continuous on the k-regular complex, it is sufficient that there is a basis b^i_{jr}, in which S has Jordan normal form, such that there is a pair of cyclic subspaces J^a_b, J^c_d in this basis, possibly coinciding, with $\lambda_a\lambda_c$ positive real, and for which the following conditions are satisfied:*

1. *For any pair of eigenbasis functions corresponding to eigenvalues λ_i and λ_k such that $|\lambda_i\lambda_k| > \lambda_a\lambda_c$ the Jacobian $J[f^i_{jr}, f^k_{lt}]$, is identically zero.*

2. *Let $\mathrm{ord}(b^i_{jr}, b^k_{lt}) = r + t$ if $J^i_j \neq J^k_l$, $\mathrm{ord}(b^i_{jr}, b^k_{lt}) = r + t - 1$ if $J^i_j = J^k_l$, and $r \neq t$. Let $\mathrm{ord}(b^i_{jt}, b^i_{jt}) = 0$. For any pair of eigenbasis functions f^i_{jr} and f^k_{lt} corresponding to eigenvalues λ_i and λ_k such that $|\lambda_i\lambda_k| = \lambda_a\lambda_c$ the Jacobian $J\left[f^i_{jr}, f^k_{lt}\right]$, is identically zero if $\mathrm{ord}(b^i_{jr}, b^k_{lt}) \geq \mathrm{ord}(b^a_{bn^a_b}, b^c_{dn^c_d})$, and $J^a_b \neq J^c_d$, or $\mathrm{ord}(b^i_{jr}, b^k_{lt}) \geq \mathrm{ord}(b^a_{bn^a_b}, b^a_{b\,n^a_b-1})$ otherwise.*

3. The Jacobian of the characteristic map of J^a_b, J^c_d has constant sign everywhere on U_1 except zero.

Condition A immediately follows from regularity of the characteristic map, and the only assumptions of this condition are C^1-continuity on the regular complex and the scaling relation.

Unlike Reif's condition, this is a condition for tangent plane continuity, rather than C^1-continuity. We discuss C^1-continuity in the next section. Our condition covers a number of cases where Reif's condition does not apply: the cyclic subspaces defining the characteristic map need not have order zero; the eigenvalues may be complex conjugate; the cyclic subspaces may coincide; the eigenvalues of the subspaces defining the characteristic map need not be subdominant.

Another condition, with stronger assumptions than the one above, but easier to check, can be obtained directly from Theorem 3.5 by relaxing the nondegeneracy assumptions; it is sufficient to assume that only the characteristic map corresponding to the dominant cyclic subspace of ΛS^T is nondegenerate.

4 Criteria for C^1 and C^k-continuity

Once tangent plane continuity is established, the only additional condition that is required for C^k-continuity is injectivity and C^k-continuity of the projection of the universal surface into the tangent plane.

This criterion for C^k-continuity can be obtained by reinterpreting the injectivity condition in terms of the eigenbasis functions. Let τ be the tangent plane, P_τ be the projection $\mathbf{R}^3 \to \tau$. Recall that $P_\tau \circ \psi$ is just the parametric map Ψ defined in Section 3.1. Suppose that arbitrarily close to zero there are points $y_1, y_2 \in U_1$, $y_1, y_2 \neq 0$, such that $\Psi(y_1) = \Psi(y_2)$. If $\psi(y_1) \neq \psi(y_2)$, the projection P_τ restricted to the tangent plane is not injective on any neighborhood of zero.

To obtain conditions for C^k-continuity we only have to note that in this case the parameterization of the universal surface over the tangent plane can be written as $\psi \circ \Psi^{-1}$ where Ψ is the parametric map. Note that Ψ can be noninjective, but conditions of Theorem 4.1 guarantee that $\psi \circ \Psi^{-1}$ is well-defined. We can replace the real eigenbasis functions which are components of ψ, with corresponding complex eigenbasis functions. Then we have the following criterion for C^k-continuity:

Theorem 4.1. *A tangent plane continuous scheme with parametric map Ψ is C^1-continuous if and only if there is a neighborhood of zero U, such that for any $y_1, y_2 \in U$, $y_1, y_2 \neq 0$ for which $\Psi(y_1) = \Psi(y_2)$, and for any eigenbasis function f, the values $f(y_1)$ and $f(y_2)$ coincide. A subdivision scheme is C^k-continuous if and only if the reparameterized eigenbasis functions $f^i_{jr}(\Psi^{-1}(\xi)) : \Psi(U) \to \mathbf{C}$ are C^k-continuous on U.*

It is important to note that the theorem does not imply that Φ is injective in any neighborhood of zero. If Ψ is injective, than $\Psi(y_1) \neq \Psi(y_2)$ for any y_1 and y_2 and the first condition of the theorem is trivially satisfied. For practically useful schemes Ψ is likely to be injective; but it is easy to construct examples when this is not the case. It is important to note that the artefacts that make surfaces with noninjective parametric maps impractical (for example, almost inevitable global self-intersections) do not preclude them from being locally C^1-continuous.

The first part of this theorem in combination with Theorem 3.6 gives a sufficient condition for C^1-continuity, which extend Reif's condition; it applies whenever Theorem 3.6 applies.

It is important to note that Reif's condition goes one step further and asserts that it is sufficient to verify regularity and injectivity of the characteristic map on an annular region. It is clear that it is sufficient to check regularity on such region due to the self-similarity of the Jacobian of the characteristic map. It is less obvious that it is sufficient to verify that the map is injective only on such region. Reif's argument also applies in our case. It is even possible to make much stronger statement: it is sufficient to verify only that the index of the characteristic map is 1, as it is described in [26] and in a separate paper [27].

Theorem 4.1 can be made much more explicit if the parametric map coincides with the characteristic map:

Condition C. *The parametric map corresponds to a pair of cyclic subspaces of order 0 with real eigenvalues, or to the cyclic subspaces of a pair of complex eigenvalues of order 0, or a single cyclic subspace of order 2. In other words, the sum of the pair of cyclic subspaces defining the parametric map has dimension 2.*

Additional motivation for considering this case is that only in this case C^1-continuity of the subdivision scheme can be *stable with respect to perturbations of coefficients of the subdivision matrix.* It is possible to show under certain assumptions that unless Condition C is satisfied, there is an arbitrary small perturbation of the entries of the subdivision matrix such that the resulting matrix violates the necessary conditions for tangent plane continuity.

Recall that there are three possible types of characteristic maps (Figure 4). The complex eigenbasis functions reparameterized by the parametric map $f^i_{jr}(\xi) = f^i_{jr}(\Psi^{-1}(\xi))$ satisfy more general scaling relations of the form

$$f^i_{jr}(T\xi) = \lambda_i f^i_{jr}(\xi) + f^i_{j\,r-1}(\xi), \quad \text{for } r \geq 1 \quad f^i_{j0}(T\xi) = \lambda_i f^r_{j0}(\xi)$$

where T is a nondegenerate linear transformation of the plane, of one of three normal forms: diagonal matrix with real eigenvalues λ_a, λ_c, rotation matrix corresponding to a pair of complex eigenvalues $\lambda\exp(i\varphi), \lambda\exp(-i\varphi)$ or a Jordan block $J_2(\lambda)$ for a real λ. We assume that $|\lambda_a|$, $|\lambda_c|$, $|\lambda| < 1$ and $\lambda_i \neq 0$. Note that any linear nondegenerate transformation of the plane can be reduced to one of these forms, so our list of transformations is exhaustive for scaling relations with linear T.

Using the results about C^k-continuity of functions satisfying scaling relations (Section 6.4), we can formulate a general criterion of C^k-continuity of subdivision schemes.

Before stating the theorem, we need to define three special types of polynomials. Each type of polynomials corresponds to a particular type of characteristic map described above.

The first two types generalize the idea of homogeneous polynomials. Their definitions differ only slightly from the standard definitions of quasihomogeneous polynomials (see for example, [1]);

1. For T being the diagonal matrix with real eigenvalues λ_a, λ_c, we use the classes of polynomials $\mathbf{P}(p,q)$. Let $\mathbf{N}(p,q)$ be the set of all pairs of non-negative integers (i,j) such that $\lambda_a^i\lambda_c^j = \lambda_a^p\lambda_c^q$ for a fixed pair (p,q). Then $\mathbf{P}(p,q)$ is defined as

$$\mathbf{P}(p,q) = \Big\{ \sum_{i,j} \alpha_{ij}\xi_1^i\xi_2^j \;\Big|\; (i,j) \in \mathbf{N}(p,q),\ \alpha_{ij} \in \mathbf{C} \Big\}$$

   Note that the set $N(p,q)$ depends on p, q and the ratio $\ln\lambda_a/\ln\lambda_c$. For example, if this ratio is $2/3$, then $\mathbf{P}(4,3)$ is spanned by the monomials ξ_1^6, $\xi_1^4\xi_2^3$, $\xi_1^2\xi_2^6$, ξ_2^{12}.

   We also define an integer constant j^i_{min} for all λ_i satisfying $|\lambda_i| \geq |\lambda_c|^k$ as

$$j^i_{min} = \min\big\{ j \;\big|\; \exists l:\ l + j \leq k, \text{ and } |\lambda_a^l\lambda_c^j| \leq |\lambda_i| \big\}$$

   Note that if $|\lambda_a| = |\lambda_c|$ and $|\lambda_i| \geq |\lambda_c^k|$, $j^i_{min} = 0$. The meaning of this constant is explained in Section 6.4.

2. If T has a pair of complex conjugate eigenvalues λ_a, $\lambda_c = \bar{\lambda_a}$, we define $\bar{\mathbf{N}}(p,q)$ as the set of all pairs of integers (i,j) such that $\lambda_a^i\bar{\lambda_a}^j = \lambda_a^p\bar{\lambda_a}^q$ for a fixed pair (p,q). In this case we define the set of polynomials

$$\bar{\mathbf{P}}(p,q) = \Big\{ \sum_{i,j} \alpha_{ij}\xi^i\bar{\xi}^j \;\Big|\; (i,j) \in \bar{\mathbf{N}}(p,q),\ \alpha_{ij} \in \mathbf{C} \Big\}$$

3. If T is a Jordan block of size 2 with real eigenvalue λ, we use polynomials

$$F_m(t) = \frac{1}{m!}\prod_{i=0}^{m-1}(x-i) \text{ for } m > 0;\ F_0(t) = 1 \tag{4.1}$$

Theorem 4.2. *Suppose that a subdivision scheme is C^k-continuous on regular complexes, tangent plane continuous at extraordinary points of a fixed valence, and the parametric map of the scheme coincides with the characteristic map Ψ. The subdivision scheme is C^1-continuous on the k-regular complex if and only if Ψ has Jacobian of constant sign, for any y_1 and y_2 for which $\Psi(y_1) = \Psi(y_2)$, and for any eigenbasis function f the values $f(y_1)$ and $f(y_2)$ coincide. The subdivision scheme is C^k-continuous if in addition, for $|\lambda_i| > |\lambda_c|^k$, any nontrivial set of complex eigenbasis functions $f^i_{jr}(\Phi^{-1}(\xi)) = f^i_{jr}(\xi)$, $r = 1..n_j$ corresponding to the eigenvalue λ_i satisfies one of the following conditions:*

1. The characteristic map corresponds to a pair of different cyclic subspaces with real eigenvalues λ_a, λ_c and

(a) $\lambda_i = \lambda_a^p \lambda_c^{q+j^i_{min}}$ *for some nonnegative* p, q , $p+q \le k - j^i_{min}$, $p+q \neq 0$ *and* $\partial_2^{j^i_{min}} f^i_{jn^i_j}(\xi) \in \mathbf{P}(p,q)$, $\partial_2^{j^i_{min}} f^i_{jm}(\xi) \equiv 0$ *for* $m < n^i_j$.

(b) OR $\partial_2^{j^i_{min}} f^i_{jr}(\xi) \equiv 0$ *for all* j.

2. *The characteristic map corresponds to a pair of complex conjugate eigenvalues* λ_a, $\bar{\lambda_a}$, $\lambda_i = \lambda_a^p \bar{\lambda_a}^q$ *for some* p, q, $p+q \le k$, $f^i_{jn^i_j}(\xi) \in \bar{\mathbf{P}}(p,q)$, *and* $f^i_{jm}(\xi) \equiv 0$ *for* $m < n^i_j$.

3. *The characteristic map corresponds to a single cyclic subspace with real eigenvalue* λ_a, $\lambda_i = \lambda_a^p$ *for some* $p \le k$ *and*

$$f^i_{jr}(\xi) = \sum_{i=0}^{r-l} C_{r-l-m} \frac{\xi_2^p}{\lambda_a^{mp}} F_m(\frac{\lambda_a \xi_1}{\xi_2})$$

for $r \ge n^i_j - p$, *where* $l = \max(0, n^i_j - p)$. *For* $r < n^i_j - p$, $f^i_{jr}(\xi) \equiv 0$.

The theorem immediately follows from Theorem 4.1 combined with the criteria of C^k-continuity of functions satisfying scaling relations stated in Section 6.4.

An important special case of Theorem 4.2 occurs when $\lambda_a = \lambda_c$; in this case the eigenvalues are necessarily real and the criterion becomes

Corollary 4.3. *If a subdivision scheme satisfies conditions of Theorem 4.2 and* $\lambda_a = \lambda_c = \lambda$, *than the scheme is* C^k *if and only if any nonzero complex eigenbasis function* $f^i_{jn^i_j}$ *corresponding to an eigenvalue* $\lambda_i \ge \lambda^k$ *is a homogeneous polynomial of degree* d, $\lambda_i = \lambda^d$ *and for all* $r < n^i_j$ $f^i_{jr} \equiv 0$.

Another important special case are the conditions for C^1-continuity:

Corollary 4.4. *If for a tangent plane continuous subdivision scheme the characteristic map* Ψ *coincides with the parametric map, it is* C^1*-continuous if and only if for any* $y_1, y_2 \in U_1$, $y_1, y_2 \neq 0$, *such that* $\Psi(y_1) = \Psi(y_2)$, *for any eigenbasis function* $f(y_1) = f(y_2)$ *and the Jacobian of* Ψ *has constant sign.*

The C^k-continuity conditions for the case when the characteristic map is (f^a_{b0}, f^c_{d0}) with λ_a, λ_c real can be made more explicit, if we integrate $\partial_2^{j^i_{min}} f^i_{jm}$; see Section 6.4.

5 Conclusion

We have presented a number of conditions for tangent plane continuity and C^k-continuity of stationary subdivision, unifying and extending most of the known conditions. Considering subdivision surfaces locally as images of the universal maps allowed us to separate the geometric and algebraic aspects of the problem. In addition, we avoid considering the degenerate configurations of control points, corresponding to special directions of projection. Our approach makes most of the arguments more intuitive compared with the more common purely analytical approach.

A number of questions remained unanswered. Ideally, tangent plane continuity and C^1-continuity conditions should be formulated purely in terms of the coefficients of subdivision schemes. Our conditions still require considering limit functions of subdivision. While it is unlikely that explicit conditions on coefficients can be established in general, it might be possible to find such conditions in special cases.

C^k-continuity criteria suggest a simple way of constructing subdivision schemes with higher degree of smoothness: we just have to ensure that the magnitude of certain the eigenvalues is sufficiently small. A C^2-subdivision scheme constructed using this approach was proposed by Prautzsch and Umlauf [21]; however, such schemes generate surfaces that are flat at extraordinary vertices (have zero curvature). At the same time degree bounds were derived for piecewise-polynomial schemes ([23, 20]), which indicate that schemes that generate non-flat surfaces of higher-order continuity are likely to have large supports. Still, construction of such schemes is of some interest.

Another important question that we have mentioned is stability with respect to small perturbations of coefficients.

From the practical point of view, it is important to develop a systematic way of applying criteria of the type derived in this paper and previously derived by other authors to specific subdivision schemes and families of subdivision schemes. We will address this issue in a separate paper [27].

One of the advantages of the approach developed in this paper is that it is possible to extend it to subdivision on higher-dimensional complexes; for example, we may consider complexes with cubic cells, which are refined by splitting each cell into eight subcells. Instead of the universal surface we would consider a three-dimensional manifold, instead of 2-vectors, 3-vectors, etc. In this way, our approach can be potentially extended to schemes of the type proposed in [16].

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6 Proofs

In this section we present several proofs that were postponed in previous sections.

6.1 Proof of Proposition 1.2

Necessity of the condition of the proposition is obvious; we prove sufficiency. Suppose P_τ is injective on $f(D)$. We are going to show that the inverse of $P_\tau|_{f(D)}$ is regular as a function on the tangent plane in a neighborhood of $f(0)$.

By assumption, there is a parameterization p of the surface defined on a neighborhood of zero in the plane, which is regular away from zero, and the limit $\lim_{y\to 0}[\partial_1 p(y) \wedge \partial_2 p(y)]_+ = u_0$ exists. Choose the basis in $\mathbf{R}^p$ in such a way that $u_0 = e_1 \wedge e_2$. As any nondegenerate projections into a plane differ by a nondegenerate affine transformation which does not affect injectivity, we can assume that $P_\tau(a) = (a^1, a^2)$ for any $a \in R^p$, if $a = \sum_i a^i e_i$. Let $p^i(y)$ be the i-th coordinate function of p. Let $\Phi(y) = (p^1(y), p^2(y))$. Note that the components of $\partial_1 p(y) \wedge \partial_2 p(y)$ are the Jacobians of the pairs of coordinate functions $J[p^i, p^j]$. As $\lim_{y\to 0}[\partial_1 p(y) \wedge \partial_2 p(y)]_+ = e_1 \wedge e_2$, for a sufficiently small neighborhood of zero U $(\partial_1 p(y) \wedge \partial_2 p(y), e_1 \wedge e_2) > 0$; but this component of $\partial_1 p(y) \wedge \partial_2 p(y)$ is exactly the Jacobian of Φ. We conclude that Φ is regular on U away from zero.

Therefore, for any point $x \in U$, $x \neq 0$, there is a neighborhood U_x of x such that Φ is invertible.

Also note that $[\partial_1 p(y) \wedge \partial_2 p(y)]_+ = \partial_1 p(y) \wedge \partial_2 p(y) / \left\|\partial_1 p(y) \wedge \partial_2 p(y)\right\|$. Writing the components of the equation $\lim_{y\to 0}[\partial_1 p(y) \wedge \partial_2 p(y)]_+ = e_1 \wedge e_2$ explicitly, we obtain

$$\lim_{y\to 0} \frac{J[\Phi]}{\left\|\partial_1 p(y) \wedge \partial_2 p(y)\right\|} = 1 \tag{6.1}$$

$$\lim_{y\to 0} \frac{J[p^i, p^k]}{\left\|\partial_1 p(y) \wedge \partial_2 p(y)\right\|} = 0 \quad \text{for } (i,k) \neq (1,2) \tag{6.2}$$

combining (6.1) with (6.2) we obtain

$$\lim_{y\to 0} \frac{J[p^i, p^k]}{J[\Phi]} = 0 \quad \text{for } (i,k) \neq (1,2)$$

Let $P_\tau|_{f(D)} = \pi$. As $\Phi = P_\tau \circ p$, then on U_x we can write $\pi = p \circ \Phi^{-1}$. Observe that $\partial_1\pi \wedge \partial_2\pi = J[\Phi]^{-1} \partial_1 p \wedge \partial_2 p$. Let M_J be the Jacobi matrix of Φ.

Note that the we can write the vector $[\frac{J[p^i,p^k]}{J[\Phi]}, \frac{J[p^i,p^k]}{J[\Phi]}]^T$ as

$$\frac{1}{J[\Phi]} \begin{bmatrix} J[p^i, p^1] \\ J[p^i, p^2] \end{bmatrix} = M_J^{-1} \begin{bmatrix} \partial_1 p^i \\ \partial_2 p^i \end{bmatrix} \tag{6.3}$$

which is exactly the gradient $[\partial_1 \pi^i \; \partial_2 \pi^i]^T$ of the i-th component of π. As we have observed, each component of the left-hand side of (6.3) converges to zero for $i \neq 1, 2$. For $i = 1, 2$ the components of π are just linear functions away from zero and their gradients have limits $[1, 0]$ and $[0, 1]$ at zero.

If the limit of a derivative exists at zero, the derivative itself exists at zero and is continuous. We conclude that π is a regular parameterization of the surface.

6.2 Proof of Lemma 3.1

Using (3.2) and (3.3) we can write an expression for $A^s v$ in terms of vectors $v_\mu^{(q)}$ (real μ) and $v_{1\mu}^{(q)}$, $v_{2\mu}^{(q)}$ (complex μ), $q = 0 \dots k_\mu$. Define $r_\mu(s,q) = |\mu|^{s-k_\mu+q} \begin{pmatrix} s \\ k_\mu - q \end{pmatrix}$. Then

$$\begin{aligned} A^s v = &\sum_{\text{real } \mu>0} \sum_{q=0}^{k_\mu} r_\mu(s,q) v_\mu^{(q)} \\ &+ \sum_{\text{real } \mu<0} \sum_{q=0}^{k_\mu} (-1)^{s-k_\mu+q} r_\mu(s,q) v_\mu^{(q)} \\ &+ \sum_{\text{complex } \mu} \sum_{q=0}^{k_\mu} r_\mu(s,q)(v_{\mu 1}^{(q)} \cos((s+q-k)\chi_\mu) - v_{\mu 2}^{(q)} \sin((s+q-k)\chi_\mu)) \end{aligned} \tag{6.4}$$

The set of vectors $v_\mu^{(q)}$, $v_{1\mu}^{(q)}$, $v_{2\mu}^{(q)}$, $\mu \in \mathcal{M}(v)$, is linearly independent. Therefore, we can construct a basis such that this set of vectors is a part of the basis. In a finite-dimensional space any basis is a Riesz basis, in particular, there is a constant B such that

$$\begin{aligned} \left\|A^s v\right\| \geq B \Bigg(&\sum_{\text{real } \mu} \sum_{q=0}^{k_\mu} r_\mu(s,q) \left\|v_\mu^{(q)}\right\| \\ &+ \sum_{\text{complex } \mu} \sum_{q=0}^{k_\mu} r_\mu(s,q)(\left\|v_{\mu 1}^{(q)}\right\| |\cos((s+q-k)\chi_\mu)| + \left\|v_{\mu 2}^{(q)}\right\| |\sin((s+q-k)\chi_\mu)|) \Bigg) \end{aligned} \tag{6.5}$$

Consider the direction of $A^s v$, that is, $A^s v / \left\|A^s\right\|$. As all components of the vector are independent, this vector has a limit if and only if each component has a limit.

Suppose $\mu \in \mathcal{M}(v)$ is complex. Define

$$v(s) = v_{1\mu}^{(0)} \cos s\chi_\mu - v_{1\mu}^{(0)} \sin s\chi_\mu$$

Intuitively it is clear that this sequence of vectors does not have a limit direction; there are two sequences s_k^1, s_k^2 such that $v(s_k^1)$ and $v(s_k^2)$ converge to linearly independent limits as $k \to \infty$. For irrational $\chi_\mu/2\pi$, this follows from the well-known fact (see for example, Hardy [12]) that for any ϵ $t \in [0, 2\pi]$ and arbitrary large s, there is an s' such that $|s\chi_\mu \bmod 2\pi - t| < \epsilon$. If $\chi_\mu/2\pi$ is rational, then the function is periodic, and unless it is constant, which is impossible, we can choose two constant subsequences of linearly independent vectors.

Let s_k^1, s_k^2 are two sequences such that $v(s_k^1)$ converges to c^1 and $v(s_k^2)$ converges to c^2 as $k \to \infty$, with c^1 and c^2 linearly independent.

Because $\mu \in \mathcal{M}(v)$, $k_\mu = k_M$ and the ratio $r_{\mu'}(s,q)/r_\mu(s,k_M)$ as $s \to \infty$ for all μ' and q. From (6.5) we have

$$\left\|A^{s_k^1} v\right\| \geq B r_\mu(s,k_M)(\left\|c^1\right\|(1-\epsilon)) \tag{6.6}$$

for arbitrary small ϵ and for sufficiently large k. Similar statement is true for s_k^2. Therefore, all elements of the sequence $A^s v / \left\|A^s v\right\|$ are well-defined for sufficiently large k.

Also from definition of M and k_M it follows that

$$\left\|A^s v\right\| < K r_M(s,k_M) \text{ for some constant } K. \tag{6.7}$$

Observe that $v(s)r_\mu(s,k_M)/\left\|A^s v\right\|$ is a linearly independent component in the decomposition of $A^s v/\left\|A^s v\right\|$. To show that $A^s v/\left\|A^s v\right\|$ does not have a limit it is sufficient to show that $v(s)r_\mu(s,k_M)/\left\|A^s v\right\|$ does not have a limit.

For sufficiently large k and arbitrarily small ϵ

$$\frac{\left\|v(s_k^1)r_M(s_k^1,k_M)\right\|}{\left\|A^{s_k^1}v\right\|} \geq \frac{1}{K}\left\|c^1\right\|(1-\epsilon)$$

The direction of the vectors in the sequence $v(s_k^1)r_\mu(s_k^1,k_M)/\left\|A^{s_k^1}v\right\|$ converges to $c^1/\left\|c^1\right\|$; the direction of the vectors in the sequence $v(s_k^2)r_\mu(s_k^2,k_M)/\left\|A^{s_k^2}v\right\|$ converges to $c^2/\left\|c^2\right\|$. By linear independence of c^1 and c^2 these limits do not coincide.

Therefore, the component does not have a limit as $s\to\infty$ and we conclude that the sequence $A^s v$ does not have a limit direction.

Similar argument can be used to show that that $\mu\in\mathcal{M}$ cannot be negative.

Thus, if the sequence of vectors has a limit direction, the eigenvalues in $\mathcal{M}(v)$ are all positive and real. But the magnitudes of all eigenvalues in $\mathcal{M}(v)$ are equal, therefore, it may contain only a single element.

Convergence rate can be easily estimated observing that the ratio of the second slowest decreasing term to the dominant term decreases at least at the rate $O(s^{-1})$.

6.3 Proof of Lemma 3.4

Complex Jordan structure of ΛB. It is straightforward to show that any eigenvalue μ of ΛB is a product of eigenvalues $\lambda_i\lambda_k$ of B (i and k may coincide.) Suppose B has cyclic subspaces J_j^i and J_l^k corresponding to eigenvalues λ_i and λ_k, of orders n_j^i and n_l^k respectively. Let e_{jr}^i,e_{lt}^k be the vectors of the Jordan basis of B with our usual conventions.

Two cases are possible: J_j^i and J_l^k are different subspaces, or they coincide. In the first case the cyclic subspaces J_j^i and J_l^k generate a subspace $J_j^i\wedge J_l^k$ of $\Lambda^2(\mathbf{R}^p)$ of dimension $(n_j^i+1)(n_l^k+1)$. In the second case, we obtain a subspace of dimension $n_j^i(n_j^i+1)/2$. Each subspace $J_j^i\wedge J_l^k$ is composed of several cyclic subspaces. It can be easily shown (Figure 5) that the maximal order of an element of $J_j^i\wedge J_l^k$ is $n_j^i+n_l^k$. Therefore, the largest cyclic subspace $\Pr\left(J_j^i\wedge J_l^k\right)$ has order $n_j^i+n_l^k$. It is easy to show that other cyclic subspaces have order strictly less then $n_j^i+n_l^k$ (in fact, the orders of other cyclic subspaces are $n_j^i+n_l^k-2m$), $m=1\ldots$. For $J_j^i\wedge J_j^i$ the order of the largest cyclic subspace is $2n_j^i-2$; if $n_j^i=0$, $J_j^i\wedge J_j^i$ is trivial.

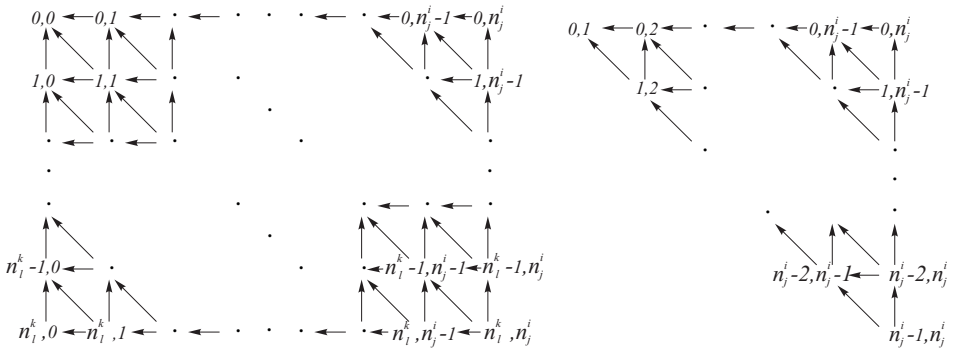

Figure 5: Left: the subspace of $\Lambda^2(\mathbf{R}^p)$ generated by two cyclic subspaces of $\mathbf{R}^p$ J_j^i, J_l^k. The pairs of numbers correspond to the basis vectors $e_{jr}^i\wedge e_{lt}^k$; arrows indicate the components that are generated by each vector after one application of $\Lambda B-\lambda_i\lambda_k I$; after m steps, if we start in the bottom right corner, we only have components above the line given by equation $r+t\leq n_j^i+n_l^k-m$. Right: Subspace of $\Lambda^2(\mathbf{R}^p)$ generated by a single cyclic subspace J_j^i. The pairs of numbers correspond to the basis vectors.

Conditions for existence of a single dominant cyclic subspace of ΛB. Recall that we call a cyclic subspace J_M of ΛB dominant, if it corresponds to a real positive eigenvalue M, and for any other cyclic subspace of order k corresponding to the eigenvalue μ, $(\mu,k)<(M,k_M)$ where k_M is the order of J_M.

We have observed that any eigenvalue of ΛB has the form $\lambda_i\lambda_k$ or λ_i^2; and the orders of cyclic subspaces are of the form $n_j^i + n_l^k - 2m$ and $2n_j^i - 2 - 4m$, $m = 0\ldots$. Therefore, we need to assert that $(M, k_M) > (\lambda_i\lambda_k, n_j^i + n_l^k)$ and $(M, k_M) > (\lambda_i^2, 2n_j^i - 2)$ for all other pairs of cyclic subspaces of B different from the pair defining to J_M.

We have to consider only the subspaces $\Pr\left(J_j^i \wedge J_l^k\right)$; other cyclic subspaces of ΛB have smaller orders. With our ordering of cyclic subspaces M can be either λ_1^2 or $\lambda_1\lambda_2$. The options for the dominant subspace are $\Pr\left(J_1^1 \wedge J_1^1\right)$, $\Pr\left(J_1^1 \wedge J_2^1\right)$, $\Pr\left(J_1^1 \wedge J_1^2\right)$ (real eigenvalues), and $\Pr\left(J_1^1 \wedge J_1^2\right)$ (complex-conjugate eigenvalues).

The first two cases require λ_1 real, the third case requires λ_1 and λ_2 real, and the last case requires λ_1 and λ_2 to be complex conjugate with $\lambda_1\lambda_2$ real positive. These four possible cases correspond to the cases of Lemma 3.4.

1. $J_M = \Pr\left(J_1^1 \wedge J_1^1\right)$; this case implies that $M = \lambda_1^2$. Therefore, λ_1 is real. In addition, we need for any i, j, $(\lambda_i\lambda_j, n_1^i + n_1^k) < (\lambda_1, 2n_1^1 - 2)$ and $(\lambda_i, 2n_1^i - 2) < (\lambda_1, 2n_1^1 - 2)$. As $|\lambda_1| \geq |\lambda_i|$ for any $i > 1$, and $n_1^1 \geq n_j^1$ for any $j > 1$, it is sufficient to require $n_1^1 > n_2^1 + 2$, and $(\lambda_1, n_1^1) > (\lambda_i, n_1^i + 2)$ for all $i > 1$.

2. $J_M = \Pr\left(J_1^1 \wedge J_2^1\right)$, $M = \lambda_1^2$ and λ_1 are real. Similarly, the additional conditions are $n_1^1 < n_2^1 + 2$, $n_2^1 > n_3^1$ if n_3^1 is defined, and $(\lambda_1, n_1^2) > (\lambda_i, n_1^i)$ for all $i > 1$.

3. $J_M = \Pr\left(J_1^1 \wedge J_1^2\right)$, $M = \lambda_1\lambda_2$, and λ_1 and λ_2 both are complex and have opposite phase. Suppose that $|\lambda_1| \neq |\lambda_2|$; then $\bar{\lambda_1} = \lambda_i$ and $\bar{\lambda_2} = \lambda_k$ are also eigenvalues of B distinct from λ_1 and λ_2. Then eigenvalue $\lambda_1\lambda_2 = \lambda_i\lambda_k$ has a cyclic subspace $\Pr\left(J_1^i \wedge J_1^k\right)$ distinct from $\Pr\left(J_1^1 \wedge J_1^2\right)$ and of the same size, because $n_1^i = n_1^1$ and $n_1^k = n_1^2$. Therefore, there is no dominant subspace unless $\lambda_1 = \bar{\lambda_2}$. Suppose $\lambda_1 = \bar{\lambda_2}$. For $\Pr\left(J_1^1 \wedge J_1^2\right)$ to be dominant, we need $n_2^1 < n_1^1$ if n_2^1 is defined and $(\lambda_i, n_1^i) < (\lambda_1, n_1^i)$ for all $i > 2$.

4. If λ_1 and λ_2 are real, for M to be positive, they have to be of the same sign. Then necessarily $|\lambda_1| > |\lambda_2|$. To guarantee that $(\lambda_1\lambda_2, n_1^1 + n_1^2) > (\lambda_i\lambda_k, n_1^i + n_1^k)$ we need $n_1^1 < n_1^2 - 2$, $n_2^2 < n_1^2$ and $(\lambda_i, n_1^i) < (\lambda_2, n_1^2 - 1)$ for all i.

6.4 Scaling relations

In this section we outline the proofs of the criteria for C^k-continuity of functions satisfying scaling relations. Importance of scaling relations for subdivision was recognized by Warren [25]. Some of the properties of quasihomogeneous functions that are discussed here, as well as the use of Newton diagrams for establishing these properties, are well-known in the singularity theory literature and first were used, to the best of our knowledge, in the work of Kushnirenko [14].

We consider the scaling relation of the form

$$f(T\xi) = J_{n+1}(\lambda')f(\xi) \tag{6.8}$$

where T is a nondegenerate linear transformation of $\mathbf{R}^2$, f is a map $\mathbf{R}^2 \to \mathbf{C}^{n+1}$, $J_{n+1}(\lambda')$ is a Jordan block with eigenvalue λ', possibly complex.

The following Lemma proved in [26] is the basis of our derivations. This lemma extends a similar lemma of Warren [25].

Lemma 6.1. *Suppose $f(\xi) = [f_n, f_{n-1}\ldots f_0]^T : \mathbf{R}^2 \to \mathbf{C}^{n+1}$ is a continuous function defined on $D \setminus \{0\}$, where D is a compact domain in $\mathbf{R}^2$ which contains the origin as an internal point and satisfies* (6.8)

1. *If $|\lambda'| < |\lambda_{min}|^k$, where λ_{min} is the eigenvalue of T with minimal absolute value, then*

$$\lim_{|\xi|\to 0} \frac{\left\|f(\xi)\right\|}{\left\|\xi_i\right\|^k} = 0$$

2. *If $\lambda' = 1$, then f is continuous at 0 if and only if $f_n = const$ and $f_m = 0$ for $m < n$.*

3. *If $|\lambda'| \geq 1$, and $\lambda' \neq 1$, then f are continuous if and only if $f \equiv 0$.*

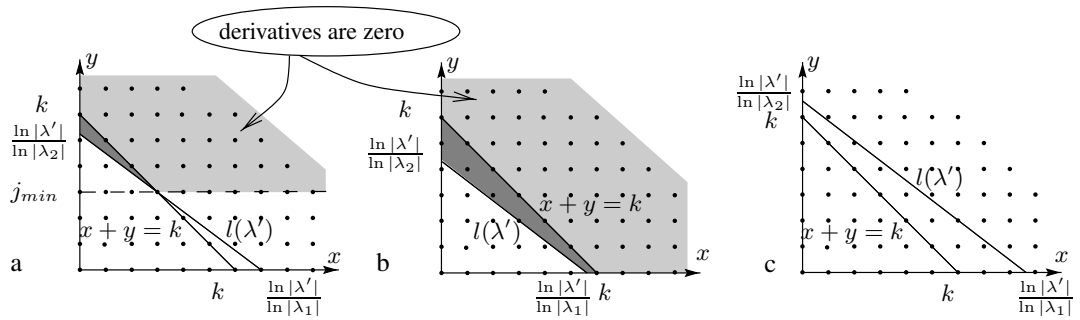

Figure 6: The Newton diagrams. (a) $|\lambda'| > |\lambda_2|^k$, $j_{min} = 0$; The function f_m has to be a polynomial. (b) $|\lambda'| > |\lambda_2|^k$, $j_{min} \neq 0$; derivative $\partial_2^{j_{min}} f_m$ has to be a polynomial. (c) $|\lambda'| < |\lambda_2|^k$; All derivatives up to order k exist.

Two real eigenvalues. We start with the case when T can be reduced to the normal form $\mathrm{diag}(\lambda_1, \lambda_2)$. This case includes the case when $\lambda_1 = \lambda_2$, and the matrix has a single real eigenvalue but with two cyclic subspaces. We assume without the loss of generality that $|\lambda_1| \geq |\lambda_2|$. We say that a system of functions $f_m(\xi)$ satisfies (λ_1, λ_2)-scaling relation for λ' if it satisfies (6.8) with $T = \mathrm{diag}(\lambda_1, \lambda_2)$.

The derivatives of functions satisfying a scaling relation do not satisfy a scaling relation themselves, but their scaled versions do. If a system of C^k-continuous functions $f_0(\xi), \dots f_m(\xi)$ satisfies the (λ_1, λ_2)-scaling relation for λ' then the derivative $\partial_1^i \partial_2^j f_m(\xi)$, $i + j \leq k$, exists at 0 and is continuous if and only if one of the following conditions is met:

1. $|\lambda'| < |\lambda_1^i \lambda_2^j|$
2. $\lambda' = \lambda_1^i \lambda_2^j$, $\partial_1^i \partial_2^j f_n(\xi) \equiv$ const and $\partial_1^i \partial_2^j f_m(\xi) \equiv 0$ for all $m < n$.
3. $\partial_1^i \partial_2^j f_m(\xi) \equiv 0$ for all m.

The functions $f_m(\xi)$ are C^k-continuous if all derivatives

$$\partial_1^i \partial_2^j f_m(\xi)$$

with $i + j \leq k$ exist and are continuous. The derivative $\partial_1^i \partial_2^j f_m$ can be associated with the integer point (i, j) in the plane. Such representation is used for the *Newton diagrams* of quasihomogeneous polynomials (see for example [1]). We are interested in the existence and continuity of the derivatives which are represented by integer points inside the triangle bounded by $x = 0$, $y = 0$, $x + y = k$ (Figure 6).

The derivatives $\partial_1^i \partial_2^j f_m$ are guaranteed to exist at 0 if $\left|\frac{\lambda'}{\lambda_1^i \lambda_2^j}\right| < 1$. Taking logarithms of both sides of this inequality, we can see that for all integer points below the line $l(\lambda')$ with equation $x \ln|\lambda_1| + y \ln|\lambda_2| = \ln|\lambda'|$, the derivatives are known to exist. For the points between the lines $l(\lambda')$ and $x + y = k$, the derivatives have to be either 0 or constants to exist and be continuous. For those that are constants, additional condition $\lambda' = \lambda_1^i \lambda_2^j$ have to be satisfied; only the derivatives of f_n can be constant; derivatives of f_m for $m < n$ are identically zero.

Note that if a derivative $\partial_1^i \partial_2^j f_m$ is 0 or constant, all derivatives to the right and upward from (i, j) are equal to zero everywhere. Suppose $|\lambda'| \geq |\lambda_2^k|$; this means that $l(\lambda')$ intersects the y axis below or at the point $(0, k)$. In this case let j_{min} be the minimal integer value of y for which there is an integer point (x, j_{min}) between $l(\lambda')$ and $x + y = k$. All derivatives represented by integer points inside the area delimited by $x = 0$, $l(\lambda')$, $x + y = k$, $y = j_{min}$ are 0 (shaded area in Figure 6).

From these considerations we obtain the following lemma:

Lemma 6.2. *All functions $f_m(\xi)$, $m = 0 \dots n$ are C^k-continuous at 0 if and only if one of the following conditions holds*

1. $|\lambda'| < |\lambda_2|^k$,
2. $\lambda' = \lambda_1^p \lambda_2^{q+j_{min}}$ *for some* p, q, $p + q \leq k - j_{min}$, $\partial_2^{j_{min}} f_n(\xi) \in \mathbf{P}(p, q)$, *and* $\partial_2^{j_{min}} f_m(\xi) \equiv 0$ *for* $m < n$.

3. $\partial_2^{j_{min}} f_m(\xi) \equiv 0$ *for all* m.

The condition on $\partial_2^{j_{min}} f_m$ does not give the explicit form for the functions f_m unless $j_{min} = 0$. It is possible to find a more explicit expression for f_m that are C^k and satisfy scaling relation for λ' (see [26]). If $\lambda' = \lambda_1^p \lambda_2^{q+j_{min}}$ for some nonnegative p, q, $p + q \le k - j_{min}$ then

$$f_n(\xi) = \xi_2^{j_{min}} p(\xi_1, \xi_2) + \sum_{s=0}^{j_{min}-1} \lambda_2^{-sn} f_n^s(\xi_1)\xi_2^s$$
$$f_m(\xi) = \sum_{s=0}^{j_{min}-1} \lambda_2^{-sm} f_m^s(\xi_1)\xi_2^s$$

where f_m^s are C^k-continuous and satisfy the λ_1-scaling relation for $\lambda_1^p \lambda_2^{j+j_{min}-s}$.

The case of T being a rotation matrix with complex-conjugate eigenvalues is similar; instead of derivatives ∂_1, ∂_2 we use $\partial = \partial_1 - i\partial_2$ and $\bar\partial = \partial_1 + i\partial_2$ to obtain the conditions on the functions f_m. The functions $f_m(\xi)$ are C^k-continuous if and only if one of the following conditions is met:

1. $|\lambda| < |\boldsymbol{\lambda}|^k$,
2. $\lambda' = \boldsymbol{\lambda}^p \bar{\boldsymbol{\lambda}}^q$ for some p, q, $f_n(\xi) \in \bar{\mathbf{P}}(p, q)$, $p + q \le k$, and $f_m(\xi) \equiv 0$ for $m < n$.
3. $f_m(\xi) \equiv 0$ for all m.

Jordan block of size 2. Finally, we consider the case when $T = J_2(\lambda)$. In this case it is convenient to consider the whole set of derivatives of the functions g_m together. Differentiated scaling relations for all m and all $\partial_1^i \partial_2^j$ for $i + j = k$ can be written in the matrix form as

$$\begin{pmatrix} B & & & & \\ & B & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & B \end{pmatrix} \begin{pmatrix} \tilde f_n(T\xi) \\ \tilde f_{n-1}(T\xi) \\ \cdot \\ \cdot \\ \tilde f_0(T\xi) \end{pmatrix} = \begin{pmatrix} \lambda' I & I & & & \\ & \lambda' I & I & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda' I \end{pmatrix} \begin{pmatrix} \tilde f_n(\xi) \\ \tilde f_{n-1}(\xi) \\ \cdot \\ \cdot \\ \tilde f_0(\xi) \end{pmatrix} \tag{6.9}$$

where $\tilde f_m(\xi) = [\partial_2^k f_m(\xi), \partial_1^2 \partial_2^{k-2} f_m(\xi), \dots \partial_1^k f_m(\xi)]^T$ and B is a triangular matrix with λ^k on the diagonal. The matrix $\mathrm{diag}(B, \dots B)$ is clearly invertible. Let $\mathcal{B} = \mathrm{diag}(B, \dots B)^{-1}\mathcal{I}$ where $\mathcal{I}$ is the matrix on the left in (6.9), let $\mathcal{B}_N$ be its Jordan form, P the matrix such that $P\mathcal{B}P^{-1} = \mathcal{B}_N$, and let $\tilde f$ be the vector $[\tilde f_n, \tilde f_{n-1} \dots \tilde f_0]^T$; then (6.9) can be written as

$$P\tilde f(T\xi) = \mathcal{B}_N P \tilde f(\xi)$$

Note that $\mathcal{B}$ is triangular with λ'/λ^k on the diagonal. Therefore, the vector $P\tilde f$ can be separated into several sets of functions satisfying scaling relations for λ'/λ^k. It follows from Lemma 6.1 that if $|\lambda'/\lambda^k| \ge 1$ the functions $f_m(\xi)$ have to be polynomials of degree no higher than k. It is easy to show that all such polynomials have to be homogeneous, and if one of them is nonzero, $\lambda' = \lambda^j$ for some $j \le k$. Observe that we can formally write for a homogeneous polynomial of degree j

$$f_m(\xi_1, \xi_2) = \frac{\xi_2^j}{\lambda^{mj}} F_m(\lambda \xi_1/\xi_2)$$

Then polynomials F_m have to satisfy scaling relations

$$F_m(t+1) = F_m(t) + F_{m-1}(t); \quad F_0(t) = C \tag{6.10}$$

It is possible to show that all solutions of these recurrences are given by functions $\sum_{i=0}^m C_{m-i} F_i(t)$, $m = 0 \dots n$, with $F_m(t)$ given by (4.1).

Lemma 6.3. *All functions $f_m(\xi)$, $m = 0 \ldots n$ in a set satisfying scaling relations with $T = J_2(\lambda)$ are C^k-continuous if and only if one of the following conditions holds:*

1. $|\lambda'| < |\lambda^k|$.
2. $\lambda' = \lambda^j$ *for* $j \leq k$ *and for* $m \geq n - j$

$$f_m(\xi) = \sum_{i=0}^{m-l} C_{m-l-i} \frac{\xi_2^j}{\lambda^{mi}} F_i(\frac{\lambda \xi_1}{\xi_2})$$

   where $F_m(t) = \frac{1}{m!} \prod_{i=0}^{m-1}(x - i)$ *for* $m > 0$, $F_0(t) = 1$ *and* $l = \max(0, n - j)$. *For* $m < n - j$ $f_m(\xi) \equiv 0$.

3. *All* $f_m(\xi) \equiv 0$.

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